Konstantin Rybnikov

Geometry of Positive Quadratic Forms over Integers

Abstract for the Combinatorics and Number Theory Seminar 2002 April 30

An integral n-polytope P is called *Delaunay* if there is an ellipsoid D, circumscribed about P, such that the only integral points lying on D, or in its interior, are the vertices of P. The ellipsoid D, and its Delaunay polytope P, is called *perfect* if E is the only ellipsoid circumscribed about P. The Gosset polytope in R^6 provides the simplest example of such an ellipsoid.

A positive quadratic form is called *perfect* if it can be reconstructed from all representations of its minumum on Z^n - {0}. Perfect forms are important to sphere packings, as all locally optimal lattice sphere packings come from perfect forms.

A *perfect ellipsoid* is an exact inhomogeneous analog of a perfect form. Quadratic forms of perfect Delaunay ellipsoids are extreme rays of L-type domains lying in the interior of the cone of metric forms. Perfect ellipsoids are in one-to-one correspondence with extreme hypermetrics, studied in analysis and the theory of finite metric spaces. Unlike for perfect forms, no theorem asserting that there are infinitely many (integral equivalence classes of) perfect Delaunay ellipsoids is known. Erdahl and I have constructed an infinite series of perfect n-ellipsoids (n > 5) that should also be Delaunay. For n = 6 our ellpsoid coincides with Gosset's. For n = 7 the Delaunay property was proven with a computer. The challenge is to prove that this infinite series is Delaunay for all n.