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Dowling Lattices, Quasigroups, and Latin Squares

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A ``quasigroup is a group without associativity or identity: what remains is unique solvability of the equation xy=z given any two of x, y, and z. Its multiplication table is a Latin square, and conversely; so if we arbitrarily rearrange the rows and columns and relabel the entries of the Latin square we always get another quasigroup, ``equivalent to the first one. Not all quasigroups are equivalent to groups: just those that are associative.

The Dowling lattices of a group are geometries of arbitrary rank (or dimension) that geometrically encode the group structure. Dowling lattices of quasigroups also exist—and they encode equivalence classes of quasigroups—but only in rank 3 (dimension 2). The reason is that higher rank implies the associative law.

Dowling lattices can be described graph-theoretically as ``biased expansions'' of complete graphs. (I will define this.) Biased expansions of arbitrary graphs therefore generalize Dowling lattices and quasigroups. Some biased expansion graphs come from groups, but I will show how to construct many that do not. A generalized quasigroup operation is represented as a circle in the biased expansion graph and an associative property as chords of that circle; if there are few enough chords (in a certain sense), then associativity may fail and the generalized quasigroup does not arise from a group.