Problem of the Week

The Problem of the Week will return in the Fall. We thank everyone who participated and we congratulate Yuqiao Huang, who is an undisputed leader this semester. For the Summer, we suggest reviewing all the problems from the Spring and working on the additional problems posted at the bottom of the provided solutions, and the following problem.

Summer Challenge (due August 31)

Fix a circle $\Gamma$. Let $T=\triangle ABC$ be a triangle inscribed in the circle $\Gamma$ which is not a right triangle and let $H$ be the orthocenter of $T$. The line $HA$ intersects the circle $\Gamma$ at $A$ and at a second point $A_1$ (which can be $A$ if the line is tangent to the circle). Likewise, the line $HB$ intersects $\Gamma$ at a second point $B_1$, and the line $HC$ intersects $\Gamma$ at a second point $C_1$. The triangle $\triangle A_1B_1C_1$ is again inscribed in $\Gamma$. We denote this triangle by $\Phi(T)$. Warning: $\Phi(T)$ can be a right triangle.

a) Show that triangles $T$ and $\Phi(T)$ are congruent if and only if either $T$ is equilateral or the angles of $T$ are $\pi/7$, $2\pi/7$, $4\pi/7$.

b) For every integer $k>0$ find the number $t_k$ of non-congruent triangles $T$ inscribed in $\Gamma$ such that $\Phi^k(T)$ and $T$ are congruent. Here $\Phi^k$ denotes the composition $\Phi \circ \Phi \circ \ldots \circ \Phi$ of $\Phi$ with itself $k$ times. Thus, according to a), we have $t_1=2$.

c) Is it true that if $\Phi^k(T)$ and $T$ are congruent then $\Phi^m(T)=T$ for some $m$?

d) Find and prove your own results about $\Phi$.

Overview

Every other Monday (starting 02/03/20), we will post a problem to encourage students (both undergraduate and graduate) to enjoy mathematics outside of the classroom and engage our mathematical community in the problem solving activity. If you have a solution and want to be a part of it, e-mail your solution to Marcin Mazur (mazur@math.binghamton.edu) by the due date. We will post solutions (from us) as well as novel solutions from participants and record the names of those who've got the most number of solutions throughout each semester.

When you submit your solutions, please provide a detailed reasoning rather than just an answer. Also, please include some short info about yourself for our records.

Previous Problems

- **Problem 7** Solution received from Yuqiao Huang.
- **Problem 6** Solution received from Yuqiao Huang.
- **Problem 5** Solution received from Yuqiao Huang, Ashton Keith, and Naftoli Kolodny.
- **Problem 4** Solved by only one participant: Ashton Keith, a freshman majoring in math.

- **Problem 3** Solved completely only by Ashton Keith, a freshman majoring in math. A solution with some details missing was also submitted by Yuqiao Huang.

- **Problem 2** Solved by only one participant: Yuqiao Huang - a freshman and a math-major.

- **Problem 1** Solved by only one participant: Yuqiao Huang - a freshman and a math-major.