Problem 4 (due Monday, April 12)

a) Let $f:\mathbb R \longrightarrow \mathbb R$ be a differentiable function such that $f(\sin x)=\sin f(x)$ for every $x\in \mathbb R$. Prove that if f is not identically zero then $\displaystyle \lim_{x\to 0} \frac{f(x)}{x}$ exists and is equal to 1 or -1.

b) Prove that there is a continuous function $f:\mathbb R \longrightarrow \mathbb R$ such that $f(\sin x)=\sin f(x)$ and $\displaystyle \lim_{x\to 0^+} \frac{f(x)}{x}$ does not exist.

Two solutions were submitted: by Paul Barber and Ashton Keith. Neither one is complete. Ashton attempts to solve part a) under additional assumption that f' is continuous at 0. While his solution has some gaps, the ideas are very nice indeed and they can be improved to a complete solution (under the additional assumption). For more details and to see complete solutions see the following link Solution.