Problem 3 (due Monday, March 6)

Let $\displaystyle s(n)=\sum_{j=1}^n {n\choose j} \frac{1}{j}$ and $\displaystyle f(n)=\frac{2^{n+1}}{n}$. Prove that \[\lim_{n\to\infty} n\left (\frac{s(n)}{f(n)}-1\right)\] exists and find its value.

The problem arose from a question my former PhD student Andrew Kelley asked me in December 2022. We received two solutions, from Prof. Vladislav Kargin and Prof. Anton Schick. Prof. Kargin's solution uses the central limit theorem and some careful estimates of the binomial coefficients. It is not hard to see that the central limit theorem is not really needed as the estimates of binomial coefficients are sufficient. Our original solution is more elementary. Then, slightly after the “deadline”, Prof. Anton Schick came with a fully probabilistic argument. For a detailed solution and additional results and problems see the following link Solution.