Problem 3 (due Monday, March 29)

Let g be a smooth function, i.e. a function which has derivatives of all orders. Recall that $g^{(k)}$ denotes the k-th derivative of g. For non-negative integers n,k, define the function $T_{n,k}(x)$ as follows: \[T_{n,k}(x)=\sum_{j=0}^n {n\choose j} (-1)^{n-j}g^{n-j}(x) (g^j)^{(k)}(x)\] (we set $g^0=1$ and $g^{(0)}=g$). For example, $T_{3,3}(x)=3g^2(x)g^{(3)}(x)-3g(x)(g^2)^{(3)}(x)+(g^3)^{(3)}(x)$.

a) Prove that $T_{n,k}(x)=0$ for any $n>k$ and any x.

b) Find a simple explicit formula for $T_{n,n}(x)$ (for example, $T_{3,3}(x)=6(g'(x))^3$. Check this!).

Four solutions were received: from Paul Barber, Yuqiao Huang, Prof. Vladislav Kargin, and Ashton Keith. Prof. Kargin submitted a beautiful solution different from our original solution. The solutions by Paul Barber, Yuqiao Huang, and Ashton Keith follow essentially the same idea as our original solution. Detailed solutions and some very nice results related to the problem are discussed in the following link Solution.