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## Chapter 14 - Section 8 - Lagrange Multipliers

### Section Overview

In this section we learn how to use the Lagrange Technique to locate extreme (maximum or minimum) values of a multivariable function subject to some constraints. As we shall see, the technique outlined is extremely similar to the method we used in Calculus I to locate extreme values of a single variable function on an interval. We will describe the technique three times, once for a three-variable function with one constraint, once for a three-variable function with two constraints, and finally for the case of an  $n$ -variable function with  $m$  constraints. We will assume throughout that our objective function is differentiable and our constraints have non-zero gradient, except for a couple examples in which we will discuss how and why the technique may fail when a constraint gradient is the zero vector.

### Three variable function, single constraint

Let  $w = f(x,y,z)$  denote our objective function and  $g(x,y,z) = k$  our constraint; that is, we seek for the extreme values attained by the function  $f$  on the level surface  $g = k$ . Suppose  $f$  does have an extreme value at a point  $P = (x_0, y_0, z_0)$  on the surface  $g = k$ . Let  $C$  be a curve with differentiable parametrization  $\mathbf{r}(t)$  that lies on the surface  $g = k$  and passes through the point  $P$ . Let  $t_0$  denote the parameter value corresponding to the point  $P$ , so  $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$ , and let  $h(t) = f(\mathbf{r}(t))$ . Now  $h$  has an extreme value at  $t_0$ , so it has a critical point there. Since  $f$  is differentiable at  $P$  and  $\mathbf{r}$  is differentiable, it follows that the composition  $h$  is differentiable at  $t_0$ , and so the critical point  $t_0$  is a root of the derivative. Hence, by the chain rule we have

$$\begin{aligned} 0 &= h'(t_0) = \frac{\partial f}{\partial x}(x_0, y_0, z_0) \frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0) \frac{dy}{dt}(t_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0) \frac{dz}{dt}(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

Thus the gradient of  $f$  at the point  $P$  is orthogonal to the line tangent to the curve  $C$  at the point  $P$ . Since this holds for all curves in the surface  $g = k$  through the point  $P$ , it follows that the gradient of  $f$  at  $P$  is parallel to the direction of the plane tangent to the surface  $g = k$  at the point  $P$ ; in other words, at the point  $P$ , if  $\nabla g \neq \mathbf{0}$  then  $\nabla f = \lambda \nabla g$  for some number  $\lambda$ .

Let us recall the technique used in Calculus I to locate extreme values of a differentiable function  $y = f(x)$  on an open interval  $I$ . If  $f$  has an extreme value at a point  $x_0 \in I$ , then  $x_0$  is a critical point of  $f$ , and since we are assuming that  $f$  is differentiable on  $I$  we must have  $f'(x_0) = 0$ . We use this result to develop our technique as follows. Assuming that  $f$  has extreme values on  $I$ , we locate them in two steps:

1. Solve  $f'(x) = 0$
2. Evaluate  $f$  at each solution  $c$  found in step 1.; largest value is the maximum, smallest is the minimum.

The Lagrange technique works in a very similar way; assuming extreme values exist, we first find critical points, then evaluate at each to determine the extreme values. There is a slight difference in our Lagrange technique though, in that it is not the critical points of  $f$  which we seek for but rather one which involves  $f$  together with our constraint  $g = k$ . Let  $\lambda$  denote a fourth variable and set  $F = f - \lambda(g - k)$ . Critical points of  $F$  are found by solving the vector equation  $\mathbf{0} = \nabla F$ , which yields the following system of

four equations with four unknowns:

$$\begin{array}{l} 0 = F_x = f_x - \lambda g_x \\ 0 = F_y = f_y - \lambda g_y \\ 0 = F_z = f_z - \lambda g_z \\ 0 = F_\lambda = g - k \end{array}$$

We see that the first three equations represent the vector equation  $\nabla f = \lambda \nabla g$  and the fourth equation is our constraint  $g = k$ . From our discussion above, we can conclude that if  $f$  has an extreme value at a point  $P = (x_0, y_0, z_0)$  on the surface  $g = k$  and if  $\nabla g \neq \mathbf{0}$  at  $P$  then there exists a number  $\lambda_0$  such that  $\nabla f = \lambda_0 \nabla g$  at  $P$  and so  $(x_0, y_0, z_0, \lambda_0)$  is a critical point of  $F$ . Therefore, assuming that  $f$  has extreme values on the surface  $g = k$  and that the gradient of  $g$  is non-zero, we can find the extreme values in two steps:

1. Solve  $\nabla \mathbf{F} = 0$ .
2. Evaluate  $f$  at the projections in  $\mathbb{R}^3$  of each solution found in step 1.; largest is maximum, smallest is minimum

Note that in step 1. above, we don't actually need the  $\lambda$  part of the solution. You may find it's values in the course of finding  $x, y,$  and  $z$ , but more often it will just be used to help relate the important variables. The points found in step 1. are in four-space, but we need points in three space to evaluate  $f$ ; specifically, we need the  $x, y, z$  part. The projection referred to above works as follows:

$$(x, y, z, \lambda) \rightarrow (x, y, z, 0) \simeq (x, y, z)$$

In words, we replace our fourth coordinate with zero, then identify the point  $(x, y, z, 0)$  in  $\mathbb{R}^4$  with the point  $(x, y, z)$  in  $\mathbb{R}^3$ .

### Example Set One

#### One

Exercise: Use the Lagrange technique to find the maximum and minimum values of the function  $f(x, y, z) = e^{xyz}$  on the ellipsoid  $2x^2 + y^2 + z^2 = 24$ .

Solution: Set  $F = e^{xyz} - \lambda(2x^2 + y^2 + z^2 - 24)$ . Then  $\mathbf{0} = \nabla F$  yields

$$\begin{array}{l} 0 = yze^{xyz} - 4x\lambda \\ 0 = xze^{xyz} - 2y\lambda \\ 0 = xye^{xyz} - 2z\lambda \\ 0 = 2x^2 + y^2 + z^2 - 24 \end{array}$$

Multiplying equation one by  $x$ , equation two by  $y$ , and equation three by  $z$ , we see that  $xyze^{xyz} = 4x^2\lambda = 2y^2\lambda = 2z^2\lambda$ . First, suppose  $\lambda \neq 0$ . Then we have  $2x^2 = y^2 = z^2$ , and so from equation four we have  $24 = 6x^2$  hence  $x = \pm 2$  and  $y = z = \pm 2\sqrt{2}$ . Now, since we multiplied equation one by  $x$ , we have to account for the possibility that we have multiplied by  $0$ . However, in this case it is clear from equations two and three that if  $x = 0$  then also  $y = z = 0$ , contradicting equation four as  $0 \neq 24$ . Hence,  $x \neq 0$ , and similarly  $y, z \neq 0$ . Now, we've found eight points which could yield extreme values, but we observe that there are only actually two possibilities; because  $f(x, y, z) = e^{xyz}$ , the four points with an even number of negative coordinates map to the maximum value of  $e^{16}$ , while the four points with an odd number of negatives map to the minimum value of  $e^{-16}$ . Now, suppose  $\lambda = 0$ . Then  $0 = xy = xz = yz$ , so at least two of  $x, y, z$  are  $0$ , and our corresponding solutions are  $(\pm 4\sqrt{3}, 0, 0)$ ,  $(0, \pm 2\sqrt{6}, 0)$ , and  $(0, 0, \pm 2\sqrt{6})$ . At each point we have  $xyz = 0$ , so  $f = e^0 = 1$ . Since  $e^{-16} < 1 < e^{16}$ , we conclude that the maximum value of  $f$  on the given ellipsoid is  $e^{16}$  and its minimum value is  $e^{-16}$ .

## Two

Exercise: Find the maximum and minimum values attained by the function  $f(x,y,z) = x^4 + y^4 + z^4$  on the unit sphere.

Solution: Since the unit sphere is the surface  $x^2 + y^2 + z^2 = 1$  our auxiliary function is  $F(x,y,z,\lambda) = x^4 + y^4 + z^4 - \lambda(x^2 + y^2 + z^2 - 1)$ , so the vector equation  $\mathbf{0} = \nabla F$  yields the system

$$\begin{array}{l} 0 = 4x^3 - 2x\lambda = 2x(2x^2 - \lambda) \\ 0 = 4y^3 - 2y\lambda = 2y(2y^2 - \lambda) \\ 0 = 4z^3 - 2z\lambda = 2z(2z^2 - \lambda) \\ 1 = x^2 + y^2 + z^2 \end{array}$$

Using a table to organize the possible cases we quickly locate the maximum value of  $\frac{1}{3}$  and the minimum value of  $\frac{1}{3}$ :

$$\begin{array}{|c|c|c|c|c|c|} \hline x & y & z & x^2 & y^2 & z^2 & f \\ \hline \neq 0 & \neq 0 & \neq 0 & & & & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \\ & 0 & \frac{1}{2} & & \frac{1}{2} & \frac{1}{2} & \\ & 0 & \frac{1}{2} & & \frac{1}{2} & \frac{1}{2} & \\ \hline \neq 0 & 0 & \frac{1}{2} & & \frac{1}{2} & \frac{1}{2} & \\ & \neq 0 & 0 & \frac{1}{2} & & \frac{1}{2} & \\ & \neq 0 & 0 & \frac{1}{2} & & \frac{1}{2} & \\ \hline \neq 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

## Three variable function, two constraints

Let  $f(x,y,z)$  be our objective function and  $g(x,y,z) = k$  and  $h(x,y,z) = \ell$  our two constraints. Let  $F = f - \lambda(g - k) - \mu(h - \ell)$ . If  $f$  has an extreme value at a point  $(x_0, y_0, z_0)$  on the intersection of the level surfaces  $g=k$  and  $h=\ell$  and both  $\nabla g$  and  $\nabla h$  are non-zero, then there exist numbers  $\lambda_0$  and  $\mu_0$  such that  $(x_0, y_0, z_0, \lambda_0, \mu_0)$  is a critical point of  $F$ . Thus, to find extreme values of  $f$  subject to the given constraints, we solve  $\mathbf{0} = \nabla F$  and check each point.

## Example Set Two

## One

Exercise: Find the extreme values of  $z$  subject to the constraints  $x^2 + y^2 = z^2$  and  $x + y + z = 24$ .

Solution: Set  $F = z - \lambda(x^2 + y^2 - z^2) - \mu(x + y + z - 24)$  and solve the vector equation  $\mathbf{0} = \nabla F$ . We get the system of equations

$$\begin{array}{l} 0 = 2x\lambda + \mu \\ 0 = 2y\lambda + \mu \\ 1 - 2z\lambda - \mu \\ z^2 = x^2 + y^2 \\ 24 = x + y + z \end{array}$$

We see if  $x = 0$  then  $\mu = 0$ , so  $y = 0$  or  $\lambda = 0$ . If  $y = 0$  then  $z = 0$  by equation four, so equation three is  $0 = 1$  a contradiction. If  $\lambda = 0$  then again we have  $0 = 1$  for equation three, a contradiction. Hence,  $x \neq 0$ . Similarly, we find  $y, z, \lambda, \mu \neq 0$ . Now the first three equations give  $x = y = \frac{-\mu}{2\lambda}$  and  $z = \frac{1 - \mu}{2\lambda}$ . Substituting into equations four and five we have  $\frac{(1 - \mu)^2}{4\lambda^2} = \frac{\mu^2}{2\lambda^2}$  and  $24 = \frac{1 - 3\mu}{2\lambda}$ , from which we find two solutions  $\mu = -1 + \sqrt{2}$  and  $2\lambda = \frac{4 - 3\sqrt{2}}{24}$  and  $\mu = -1 - \sqrt{2}$  and  $2\lambda = \frac{4 + 3\sqrt{2}}{24}$  and so we have our minimum value  $z = -24(1 + \sqrt{2})$  when  $\mu = -1 + \sqrt{2}$  and our maximum value  $z = 24(1 + \sqrt{2})$  when  $\mu = -1 - \sqrt{2}$ .

## $n$ variable function, $m$ constraints

As usual we form an auxiliary function  $F$  with of objective and all constraints and seek for solutions to the vector equation  $\mathbf{0} = \nabla F$ . Let  $x_1, x_2, \dots, x_n$  denote the  $n$ -variables and  $g_1 = k_1, g_2 = k_2, \dots, g_m = k_m$  the  $m$  constraints. Then we introduce  $m$  variables  $\lambda_1, \lambda_2, \dots, \lambda_m$  and set  $F = f - \sum_{i=1}^m \lambda_i (g_i - k_i)$

### Example Set Three

One

Exercise: Find the maximum and minimum values attained by summing the coordinates of a point on the unit  $n$ -sphere.

Solution: Our objective function is  $f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \dots + x_n$  and our constraint is  $x_1^2 + x_2^2 + \dots + x_n^2 = 1$ , so we have  $F = x_1 + x_2 + \dots + x_n - \lambda (x_1^2 + x_2^2 + \dots + x_n^2 - 1)$ . We see that for each  $1 \leq i \leq n$ , we have  $\frac{\partial F}{\partial x_i} = 1 - 2\lambda x_i$ , and so the vector equation  $\mathbf{0} = \nabla F$  yields  $x_i = \frac{1}{2\lambda}$  for all  $i$ . Hence, from our last equation we have  $1 = n \left( \frac{1}{4\lambda^2} \right)$  or  $x_i = \frac{1}{2\lambda} = \pm \frac{1}{\sqrt{n}}$ . Thus, our maximum value is  $\sqrt{n}$ , while our minimum value is  $-\sqrt{n}$ .

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## Discussion

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