Math 504 - Earlier Homework

$\newcommand{\aut}{\textrm{Aut}} \newcommand{\end}{\textrm{End}} \newcommand{\sub}{\textrm{Sub}} \newcommand{\min}{\textrm{min}} \newcommand{\lub}{\textrm{l.u.b.}} \newcommand{\glb}{\textrm{g.l.b.}} \newcommand{\join}{\vee} \newcommand{\bigjoin}{\bigvee} \newcommand{\meet}{\wedge} \newcommand{\bigmeet}{\bigwedge} \newcommand{\normaleq}{\unlhd} \newcommand{\normal}{\lhd} \newcommand{\union}{\cup} \newcommand{\intersection}{\cap} \newcommand{\bigunion}{\bigcup} \newcommand{\bigintersection}{\bigcap} \newcommand{\sq}[2][\ ]{\sqrt[#1]{#2\,}} \newcommand{\pbr}[1]{\langle #1\rangle} \newcommand{\ds}{\displaystyle} \newcommand{\C}{\mathbb{C}} \newcommand{\R}{\mathbb{R}} \newcommand{\Q}{\mathbb{Q}} \newcommand{\Z}{\mathbb{Z}} \newcommand{\N}{\mathbb{N}} \newcommand{\A}{\mathbb{A}} \newcommand{\F}{\mathbb{F}} \newcommand{\T}{\mathbb{T}} \newcommand{\ol}[1]{\overline{#1}} \newcommand{\ul}[1]{\underline{#1}} \newcommand{\imp}{\Rightarrow} \newcommand{\rimp}{\Leftarrow} \newcommand{\pinfty}{1/p^\infty} \newcommand{\power}{\mathcal{P}} \newcommand{\calL}{\mathcal{L}} \newcommand{\calC}{\mathcal{C}} \newcommand{\calN}{\mathcal{N}} \newcommand{\calB}{\mathcal{B}} \newcommand{\calF}{\mathcal{F}} \newcommand{\calR}{\mathcal{R}} \newcommand{\calS}{\mathcal{S}} \newcommand{\calU}{\mathcal{U}} \newcommand{\calT}{\mathcal{T}} \newcommand{\gal}{\textrm{Gal}} \newcommand{\isom}{\approx} \renewcommand{\hom}{\textrm{Hom}} $

Problem Set 6 Due 04/14/2020 (complete)

1. Show that every $\varphi\in\aut_K\left(\ol{K}\right)$ induces a complete lattice automorphism on $\sub_K\left(\ol{K}\right)$. (This is part of Prop. 4.6.3 in the posted class notes)
2. Let E/K be an algebraic extension. Prove that the normal closure of E/K is the spliting field of the set of polynomials \[A = \left\{{\ds\min_K(\alpha)}\mid\alpha\in E^\times\right\}. \] (This is Prop. 4.7.3 in the posted class notes)
3. Let E/K be an infinite separable extension. Prove that $[E:K]_s =_f [E:K]$, meaning both are finite and equal, or both are infinite. (Note that this and its converse were already proved for finite extensions as Prop. 3.71.3 in the posted class notes)
4. Find an example of an algebraic extension for which $[E:K]_s =_f [E:K]$, but E/K is NOT separable.

Problem Set 5 Due 03/24/2020 (complete)

1. Let $K\leq E\leq F$, and $\alpha\in F$, algebraic over K. Prove:
   I. If α is separable over K, then it is separable over E.
   II. If α is separable over E, and E/K is separable, then α is separable over K.
2. Let $K=\F_2(s,t)$ be the field of rational functions in two variables s and t, over the two element field, $\F_2$. Let $\alpha=\sqrt{s}$ and $\beta=\sqrt{t}$, i.e. α is a root of $x^2-s\in K[x]$, and similarly for β. Prove or disprove that $K(\alpha,\beta)$ is a simple extension of K.
3. Let K be a field of characteristic p.
   I. Show that $K=K^{1/p}$ iff K is perfect.
   II. Show that the field K^{1/p^∞} is a perfect field, and the smallest perfect field that contains K.
4. Let K be a field of characteristic p. Is $\ol{K}$ separable over K^{1/p^∞}? Prove or disprove.

Problem Set 4 Due 03/10/2020(complete)

1. Show that the algebraic closure is a *closure operator*, i.e.
   I. $K\leq\ol{K}$,

   II. $\ol{\ol{K}} =\ol{K}$,
   III. $K\leq E \imp \ol{K}\leq\ol{E}$.
2. Let $\ol{K}$ be an algebraic closure of K. Show:
   I. $\ol{K}$ is minimal with the property of being an extension of K which is algebraically closed.
   II. $\ol{K}$ is maximal with the property of being an algebraic extension of K.
3. Let $\ f(x)\in K[x]$. Prove that if α is a root of $\ f(x)\ $ with multiplicity m, then α is a root of $\ f^{(i)}(x)\ $ for all $\ 0\leq i < m$.
4. Prove that if K is a perfect field, and F/K is an algebraic extension, then F is a perfect field.

Problem Set 3 Due 02/25/2020 (complete)

1. Prove the corollary stated in class: If $K\leq E_i\leq F$ and each E_i/K is algebraic, then the join $\displaystyle\bigjoin_{i\in I}E_i$ is algebraic over K.
2. Let F/K be a finite extension. Prove that $\end_K(F)=\aut_K(F)$, i.e. every endomorphism of F that fixes K is an automorphism of F.
3. Consider the extension $F=\Q(\alpha,\omega)$ of $\Q$ discussed in class, where α is a root of x^3-2 and ω is a root of x^2+x+1. Construct several automorphisms of F. Is there a bound for the number of automorphisms of F?
4. Let F/K be a field extension, and $\varphi:F\to L$ a field homomorphism. Let $\widehat{F}=\varphi(F)$ and $\widehat{K}=\varphi(K)$. Prove:
   I. $[\widehat{F}:\widehat{K}]=[F:K]$.
   II. If F/K is algebraic, then so is $\widehat{F}/\widehat{K}$.
   III. If F/K is transcendental, then so is $\widehat{F}/\widehat{K}$.
   IV. If F is an algebraic closure of K, then $\widehat{F}$ is an algebraic closure of $\widehat{K}$.

Problem Set 02 Due 02/13/2020 (complete)

1. Let F be a field extension of K, and $a_1,a_2,\dots,a_n\in F$. Prove:
   I. $ K[a_1,a_2,\dots,a_n] = K[a_1][a_2]\cdots[a_n]$, and
   II. $ K(a_1,a_2,\dots,a_n) = K(a_1)(a_2)\cdots(a_n)$.
2. Show that $\Q(\sq{2})\neq\Q(\sq{3})$. Generalize.
3. Grillet, Page 163, IV.2.1
4. Grillet, Page 163, IV.2.2, IV.2.4

Problem Set 01 Due 02/04/2020 (complete)

1. Let G be a group and $N\normaleq G$. G is solvable iff N and G/N are solvable. In this case, $$l(G) \leq l(N) + l(G/N).$$
2. If L is a poset in which every subset has a l.u.b., then every subset of L also has a g.l.b.
3. Given a lattice $(L,\join,\meet)$ in the algebraic sense, show that the binary relation $\leq$, defined by $$ x \leq y \quad iff \quad x \meet y = x, $$ is a partial order on L. Moreover, for any $x,y \in L$, $x \meet y$ is the $\glb\{x,y\}$, and $x \join y$ is the $\lub\{x,y\}$.
4. Let A be a universal algebra, and $\sub(A)$ the complete lattice of subuniverses of A. If $D\subseteq\sub(A)$ is directed, then $\ds\left(\bigunion_{X\in D}X\right)\in\sub(A)$.

Homework