Math 504 - Homework

- LaTeX-ed solutions are encouraged and appreciated.
- If you use LaTeX, hand-in a printed version of your homework.
- You are encouraged to discuss homework problems with classmates, but such discussions should NOT include the exchange or written material.
- Writing of homework problems should be done on an individual basis.
- Outside references for material used in the solution of homework problems should be fully disclosed.
- References to results from the textbook and/or class notes should also be included.
- The following lists should be considered partial and tentative lists until the word complete appears next to it.
- Use 8.5in x 11in paper with smooth borders. Write your name on top of each page. Staple all pages.

$\newcommand{\aut}{\textrm{Aut}} \newcommand{\end}{\textrm{End}} \newcommand{\sub}{\textrm{Sub}} \newcommand{\min}{\textrm{min}} \newcommand{\lub}{\textrm{l.u.b.}} \newcommand{\glb}{\textrm{g.l.b.}} \newcommand{\join}{\vee} \newcommand{\bigjoin}{\bigvee} \newcommand{\meet}{\wedge} \newcommand{\bigmeet}{\bigwedge} \newcommand{\normaleq}{\unlhd} \newcommand{\normal}{\lhd} \newcommand{\union}{\cup} \newcommand{\intersection}{\cap} \newcommand{\bigunion}{\bigcup} \newcommand{\bigintersection}{\bigcap} \newcommand{\sq}[2][\ ]{\sqrt[#1]{#2\,}} \newcommand{\pbr}[1]{\langle #1\rangle} \newcommand{\ds}{\displaystyle} \newcommand{\C}{\mathbb{C}} \newcommand{\R}{\mathbb{R}} \newcommand{\Q}{\mathbb{Q}} \newcommand{\Z}{\mathbb{Z}} \newcommand{\N}{\mathbb{N}} \newcommand{\A}{\mathbb{A}} \newcommand{\F}{\mathbb{F}} \newcommand{\T}{\mathbb{T}} \newcommand{\ol}[1]{\overline{#1}} \newcommand{\ul}[1]{\underline{#1}} \newcommand{\imp}{\Rightarrow} \newcommand{\rimp}{\Leftarrow} \newcommand{\pinfty}{1/p^\infty} \newcommand{\power}{\mathcal{P}} \newcommand{\calL}{\mathcal{L}} \newcommand{\calC}{\mathcal{C}} \newcommand{\calN}{\mathcal{N}} \newcommand{\calB}{\mathcal{B}} \newcommand{\calF}{\mathcal{F}} \newcommand{\calR}{\mathcal{R}} \newcommand{\calS}{\mathcal{S}} \newcommand{\calU}{\mathcal{U}} \newcommand{\calT}{\mathcal{T}} \newcommand{\gal}{\textrm{Gal}} \newcommand{\isom}{\approx} \renewcommand{\hom}{\textrm{Hom}} $

Problem Set 9 Due 05/05/2020 (complete)

1. Prove that a finite group G is solvable iff there is a finite sequence of subgroups \[1=H_0\leq H_1 \leq \cdots \leq H_{n-1} \leq H_n=G \] such that each $H_i\normaleq H_{i+1}$ and H_{i+1}/H_i is cyclic. Show, with a counterexample, that this equivalence does not hold in general for arbitrary groups.
2. Show that the class of solvable groups is not closed under arbitrary products.
3. (Optional) Redo Exercise 4.6.1 in the class notes (page 102)
4. Let p be prime, and $G\leq S_p$. Show that if G contains a p-cycle and a transposition, the $G=S_p$.

Problem Set 8 Due 04/28/2020 (complete)

1. Prove Theorem 4.24.1,2 in the class notes (page 90).
2. Exercise 4.6.1 in the class notes (page 101)
3. Let K and L be fields. Show that the set $\hom(K,L)$ of all homomorphisms from K to L, is linearly independent over L as a subset of the vector space L^K of all functions from K to L. In particular $\aut(K)$ is linearly independent over K.

4. Let F/K be a finite extension, and L/K its normal closure. Show that L/K is also a finite extension. Hint: if you write $E=K(\alpha_1,\dots,\alpha_n)$, and let $f_i(x)=\min_K(\alpha_i)$, show that L is the splitting field of the set $A=\{f_1(x),\dots,f_n(x)\}$.

Problem Set 7 Due 04/16/2020 (complete)

1. Prove or disprove: all cyclotomic polynomials have all their coefficients in $\{0,\pm 1\}$.
2. Show that if n is even then \[\phi_{2n}(x) = \phi_n(x^2), \] and if $n\geq 3$ is odd then \[\phi_{2n}(x) = \phi_n(-x). \]
3. Let P be a locally finite poset. For $y\neq x\in P$, show that \[\sum_{y\leq \ul{z}\leq x}\mu(z,x)=0 \] Hint: Fix $y\in P$, and then use induction on the Artinian poset \[\{u\in P\mid u > y\}. \]
4. Show that the sequence of coefficients of the cyclotomic polynomial $\phi_n(x)$, for $n\geq 2$, is palindrome, i.e. if \[\Phi_n(x)=\sum_{i=0}^{\varphi(n)}a_ix^i, \] then $a_{\varphi(n)-i}=a_i$.

Earlier Homework