Math 504 - Homework 2017

$\newcommand{\aut}{\textrm{Aut}} \newcommand{\end}{\textrm{End}} \newcommand{\sub}{\textrm{Sub}} \newcommand{\min}{\textrm{min}} \newcommand{\lub}{\textrm{l.u.b.}} \newcommand{\glb}{\textrm{g.l.b.}} \newcommand{\join}{\vee} \newcommand{\bigjoin}{\bigvee} \newcommand{\meet}{\wedge} \newcommand{\bigmeet}{\bigwedge} \newcommand{\normaleq}{\unlhd} \newcommand{\normal}{\lhd} \newcommand{\union}{\cup} \newcommand{\intersection}{\cap} \newcommand{\bigunion}{\bigcup} \newcommand{\bigintersection}{\bigcap} \newcommand{\sq}[2][\ ]{\sqrt[#1]{#2\,}} \newcommand{\pbr}[1]{\langle #1\rangle} \newcommand{\ds}{\displaystyle} \newcommand{\C}{\mathbb{C}} \newcommand{\R}{\mathbb{R}} \newcommand{\Q}{\mathbb{Q}} \newcommand{\Z}{\mathbb{Z}} \newcommand{\N}{\mathbb{N}} \newcommand{\A}{\mathbb{A}} \newcommand{\F}{\mathbb{F}} \newcommand{\T}{\mathbb{T}} \newcommand{\ol}[1]{\overline{#1}} \newcommand{\ul}[1]{\underline{#1}} \newcommand{\imp}{\Rightarrow} \newcommand{\rimp}{\Leftarrow} \newcommand{\pinfty}{1/p^\infty} \newcommand{\power}{\mathcal{P}} \newcommand{\calL}{\mathcal{L}} \newcommand{\calC}{\mathcal{C}} \newcommand{\calN}{\mathcal{N}} \newcommand{\calB}{\mathcal{B}} \newcommand{\calF}{\mathcal{F}} \newcommand{\calR}{\mathcal{R}} \newcommand{\calS}{\mathcal{S}} \newcommand{\calU}{\mathcal{U}} \newcommand{\calT}{\mathcal{T}} \newcommand{\gal}{\textrm{Gal}} \newcommand{\isom}{\approx} \renewcommand{\hom}{\textrm{Hom}} $

Problem Set 10 Due 05/08/2017 (complete)

1. Complete the proof of the following proposition. The first part was done in class. If E/K is a Galois extension, and F/K is any field extension, then EF/F is a Galois extension. Moreover, $\gal(EF/F)$ embeds in $\gal(E/K)$, and when $E/E\intersection F$ is a finite extension, \[\gal(EF/F) \isom \gal(E/E\intersection F).\]
2. Show that a finite group G is solvable iff there is a finite sequence of subgroups \[1=H_0\leq H_1 \leq \cdots \leq H_{n-1} \leq H_n=G \] such that each $H_i\normaleq H_{i+1}$ and H_{i+1}/H_i is cyclic.
3. Show, by counterexample, that the **finite** hypothesis in the previous problem is necessary.
4. Let l be a constructible straight line, A a constructible point on l, and θ a constructible angle. Show that the straight line that goes through A and forms an angle θ with l is constructible.

Problem Set 9 Due 05/01/2017 (complete)

1. Prove that a directed union of algebraically independent sets over K is algebraically independent over K. In particular, the union of a chain of algebraically independent sets over K is algebraically independent over K.
2. Given $S\subseteq T$ with S algebraically independent over K and F algebraic over $K(T)$, there is a transcendence basis B with $S\subseteq B\subseteq T$. In particular, any field extension F/K has a transcendence basis.
3. Prove the following version of the exchange property: Let F/K be a field extension, $S,T\subseteq F$ be each algebraically independent over K, with $|S| < |T|$. There is $\beta\in T-S$ such that $S\union \{\beta\}$ is algebraically independent over K.
4. Prove that for a tower $L/F/K$, \[tr.d._K(L) = tr.d._F(L) + tr.d._K(F) \]
5. Prove that if $f(t_1,\dots,t_n)$ is a symmetric **polynomial** in variables $t_1,\dots,t_n$, there exists a **polynomial** g such that $f(t_1,\dots,t_n)=g(s_1,\dots,s_n)$.

Problem Set 8 Due 04/21/2017 (complete)

1. Let G be a group, and $H_1,H_2\leq G$. Show that \[[G:H_1\intersection H_2] \leq [G:H_1][G:H_2]. \]
2. Prove that the normal closure of a finite separable extension over K is a finite Galois extension.
3. Given a projective system $(G_i|i\in I)$ of groups, with maps $(\rho_{i,j}|i\leq j)$, show that the following subset of the product \[\left\{a\in\prod_{i\in I}G_i\middle|\rho_{i,j}(a_j)=a_i \text{ for all } i\leq j\right\}, \] together with the projections on the factors is a projective limit for the system.

Problem Set 7 Due 04/07/2017 (complete)

1. Prove or disprove: the lattice of centralizers in a group G is a sublattice of $\sub(G)$, the lattice of subgroups of G.
2. What is the lattice of subgroups of U_n? What is the lattice of subfields of the cyclotomic extension $\Q(\xi_n)$? Write down the bijection between these two lattices.
3. Show that σ_S, as defined in class on 03/31/17, is an automorphism of $F/\Q$.
4. Show that the Galois group G in McCarthy's Example is isomorphic to $\power(R)$, the power set of R, with symmetric difference as the binary operation.
5. Let G be a group with identity element e. Let $\calB_e$ be a collection of subgroups of G which form a basis for the neighborhoods of e. Show that the collection \[\{gH|\ g\in G, H\in\calB_e\}, \] of all left cosets of the subgroups in $\calB_e$ is a basis for a topology on G.

Problem Set 6 Due 03/24/2017 (complete)

1. Show that if E/K is separable then $\mbox{$[E:K]_s =_f [E:K]$}$, where $=_f$ means both sides are finite and equal, or both are infinite. Note that this and its converse were proved in class for finite extensions. Show that the converse is not true in general.
2. Prove or disprove: all cyclotomic polynomials have all their coefficients in $\{0,\pm 1\}$.
3. Let P be a locally finite poset, and $x\neq y\in P$. Show that \[\sum_{y\leq z\leq x}\mu(y,z)=0 \]

Problem Set 5 Due 03/10/2017 (complete)

1. Let $K\leq E\leq F$, and $\alpha\in F$, algebraic over K. Prove:
   I. If α is separable over K, then it is separable over E.
   II. If α is separable over E, and E/K is separable, then α is separable over K.
2. Let S be a set, and $P(x,B)$ denote a property, where $x \in S$ and $B ⊆ S$. When $P(x,B)$ is true, we will say that x has the property P, with respect to B. For $A,B ⊆ S$, write $P(A,B)$ provided all elements of A have property P w.r.t. B, i.e. for all $x∈A$, we have $P(x,B)$. Let $$B^P := \{x\in S\ |\ P(x,B)\}$$ be the set of elements of S related to B via the property P. Assume the property P satisfies:
   I. All elements of B satisfy property P w.r.t. B, i.e. $x ∈ B ⇒ P(x,B)$,
   II. if x has property P w.r.t. B, and $B ⊆ A$, then x has property P w.r.t. A, i.e. $(B ⊆ A \textrm{ and } P(x,B))⇒P(x,A)$,
   III. if x has property P w.r.t. A, and $P(A,B)$, then x has property P w.r.t. B, i.e. $P(x,A) \textrm{ and } P(A,B) ⇒ P(x,B)$.
   Show that the map $B \mapsto B^P$ is a closure operator.
3. Let E/K be an algebraic extension, and let $E_i=E\intersection K^{\pinfty}$. Prove or disprove that E/E_i is separable.
4. Each $\varphi\in\aut_K(\ol{K})$ induces a complete lattice automorphism of $\sub_K(\ol{K})$. All normal extensions of K are fixed points of this automorphism.

Problem Set 4 Due 02/24/2017 (complete)

1. Find a field K of characteristic 3, and an irreducible polynomial $p(x)\in K[x]$, such that $p(x)$ is inseparable. What are the multiplicities of each of the roots of $p(x)$?
2. Let K be a field of characteristic 0, $f(x)\in K[x]$, α an element of some extension of K, and $m\in\N$. Show that the multiplicity of α as a root of $f(x)$ is $\geq m$ iff α is a root of $f^{(i)}(x)$ for all $0\leq i < m$.
3. Show that a finite subgroup of the multiplicative group $K^{\times}$ of any field K is cyclic.
4. If K is a perfect field, and F/K is an algebraic extension, then F is a perfect field.

Problem Set 3 Due 02/17/2017 (complete)

1. Let F/K be a field extension and $E,L\in\sub_K(F)$. Show that if E/K is algebraic then EL is algebraic over L. If EL is algebraic over L, does it follow that E is algebraic over K? How about $E/(E\intersection L)$?
2. Let F/K be a field extension, and $\varphi:F\to L$ a field homomorphism. Let $\widehat{F}=\varphi(F)$ and $\widehat{K}=\varphi(K)$. Prove:
   I. $[\widehat{F}:\widehat{K}]=[F:K]$.
   II. If F/K is algebraic, then so is $\widehat{F}/\widehat{K}$.
   III. If F/K is transcendental, then so is $\widehat{F}/\widehat{K}$.
   IV. If F is an algebraic closure of K, then $\widehat{F}$ is an algebraic closure of $\widehat{K}$.
3. Show that the algebraic closure is a *closure operator*, i.e.
   I. $K\leq\ol{K}$,
   II. $\ol{\ol{K}} =\ol{K}$,
   III. $K\leq E \imp \ol{K}\leq\ol{E}$.

Problem Set 2 Due 02/03/2017 (complete)

1. Let A be a universal algebra, and $\sub(A)$ the complete lattice of subalgebras of A. If $D\subseteq\sub(A)$ is directed, then $\ds\left(\bigunion_{X\in D}X\right)\in\sub(A)$.
2. Show that the direct (cartesian) product of two fields is never a field.
3. Show that $\Q(\sq{2})\neq\Q(\sq{3})$. Generalize.
4. Page 163, IV.2.1
5. Page 163, IV.2.2,4

Problem Set 1 Due 01/27/2017 (complete)

1. Let G be a group and $N\normaleq G$. G is solvable iff N and G/N are solvable. In this case, $$l(G) \leq l(N) + l(G/N)$$.
2. Given a lattice $(L,\meet,\join)$ in the algebraic sense, show that the binary relation defined by $$ x \leq y \quad iff \quad x \meet y = x $$ is a partial order on L, and for $x,y \in L$, $x \meet y$ is the g.l.b.{x,y}, and $x \join y$ is the l.u.b.{x,y}.
3. If L is a poset in which every subset has a l.u.b., then every subset of L also has a g.l.b.

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