

## Math 402 - 01 Homework (Spring 2019)

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\newcommand{\aut}{\textrm{Aut}} \newcommand{\inn}{\textrm{Inn}} \newcommand{\sub}{\textrm{Sub}}
\newcommand{\cl}{\textrm{cl}} \newcommand{\join}{\vee} \newcommand{\bigjoin}{\bigvee}
\newcommand{\meet}{\wedge} \newcommand{\bigmeet}{\bigwedge} \newcommand{\normaleq}{\unlhd}
\newcommand{\normal}{\lhd} \newcommand{\union}{\cup} \newcommand{\intersection}{\cap}
\newcommand{\bigunion}{\bigcup} \newcommand{\bigintersection}{\bigcap} \newcommand{\sq}[2][\
]{\sqrt[#1]{#2},} \newcommand{\pbr}[1]{\langle #1\rangle} \newcommand{\ds}{\displaystyle}
\newcommand{\C}{\mathbb{C}} \newcommand{\R}{\mathbb{R}} \newcommand{\Q}{\mathbb{Q}}
\newcommand{\Z}{\mathbb{Z}} \newcommand{\N}{\mathbb{N}} \newcommand{\A}{\mathbb{A}}
\newcommand{\F}{\mathbb{F}} \newcommand{\T}{\mathbb{T}} \newcommand{\ol}[1]{\overline{#1}}
\newcommand{\imp}{\Rightarrow} \newcommand{\rimp}{\Leftarrow} \newcommand{\pinfty}{1/p^{\infty}}
\newcommand{\power}{\mathcal{P}} \newcommand{\call}{\mathcal{L}} \newcommand{\calC}{\mathcal{C}}
\newcommand{\calN}{\mathcal{N}} \newcommand{\calB}{\mathcal{B}} \newcommand{\calF}{\mathcal{F}}
\newcommand{\calR}{\mathcal{R}} \newcommand{\calS}{\mathcal{S}} \newcommand{\calU}{\mathcal{U}}
\newcommand{\calT}{\mathcal{T}} \newcommand{\gal}{\textrm{Gal}} \newcommand{\isom}{\approx}
\newcommand{\idl}{\textrm{Idl}} \newcommand{\lub}{\textrm{lub}} \newcommand{\glb}{\textrm{glb}}
\newcommand{\cis}{\textrm{cis}} $

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### Problem Set 10 (complete) Due: 05/10/2019

- Let  $F$  be a field of characteristic zero,  $a \in F$ , and  $\xi = \xi_n$  a primitive  $n$ -th root of unity.
  - Show by example that  $\text{Gal}_F(F(\xi))$  need not be all of  $U_n$ .
  - Show by example that  $\text{Gal}_{F(\xi)}(F(\xi, \sqrt[n]{a}))$  need not be all of  $C_n$ .
- Let  $G$  and  $H$  be solvable groups. Prove that  $G \times H$  is solvable.
- Show that the change of variable  $y = x + (a/3)$  transforms the general cubic equation  $[x^3 + ax^2 + bx + c = 0]$  into a depressed cubic. Therefore, Cardano's formula is useful to solve any cubic equation.

### Problem Set 09 (complete) Due: 05/03/2019 Board presentation: 05/10/2019

- Prove that the homomorphism  $[\begin{array}{rccc} \psi: & U_n & \to & \text{Gal}(\mathbb{Q}(\xi_n)/\mathbb{Q}) \\ & & & \cong & \text{Gal}(\mathbb{Q}(\xi_n)/\mathbb{Q}) \end{array}]$  is surjective and injective.
- Let  $\xi_{15} = \text{cis}(2\pi/15)$  be a primitive 15th root of unity.
  - Find the group  $\text{Gal}(\mathbb{Q}(\xi_{15})/\mathbb{Q})$  and draw its lattice of subgroups.
  - Find and draw the lattice of intermediate fields of the extension  $\mathbb{Q}(\xi_{15})/\mathbb{Q}$ .
  - Write down the correspondence between the subgroups in part 1, and the subfields in part 2, using the Fundamental Theorem of Galois Theory.
- Show that any non-abelian simple group is non-solvable.
- Show that if  $d$  is a divisor of  $n$  then  $\mathbb{Q}(\xi_d)$  is a subfield of  $\mathbb{Q}(\xi_n)$ . Conclude that  $\varphi(d)$  divides  $\varphi(n)$ , and  $U_d$  is a quotient of  $U_n$ .

### Problem Set 08 (complete) Due: 04/26/2019 Board presentation: 05/03/2019

- Prove the following corollary to the Fundamental Theorem of Galois Theory. Use only the FTGT statements to prove it. Let  $E/F$  be a (finite) Galois extension, with Galois group  $G = \text{Gal}_F(E)$ . Let  $L_1, L_2 \in \text{sub}_F(E)$  and  $H_1, H_2 \in \text{sub}(G)$ .

- I.  $(L_1 \vee L_2)^* = L_1^* \vee L_2^*$
  - II.  $(L_1 \wedge L_2)^* = L_1^* \wedge L_2^*$
  - III.  $(H_1 \vee H_2)^* = H_1^* \vee H_2^*$
  - IV.  $(H_1 \wedge H_2)^* = H_1^* \wedge H_2^*$
2. Let  $f(x) \in \mathbb{Q}[x]$  be such that it has a non-real root. Let  $E$  be the splitting field of  $f(x)$  over  $\mathbb{Q}$ . Prove that  $[\mathbb{Q}(E) : \mathbb{Q}]$  has even order.
  3. Consider the polynomial  $f(x) = x^3 + 2x^2 + 2x + 2 \in \mathbb{Q}[x]$ , and  $E$  its splitting field over  $\mathbb{Q}$ .
    - I. Show that  $f(x)$  has exactly one real root. (Hint: use calculus)
    - II. Show that  $f(x)$  is irreducible over  $\mathbb{Q}$ .
    - III. Find  $[E : \mathbb{Q}]$ . Fully explain your calculation.
    - IV. Determine  $[\text{Gal}(\mathbb{Q}(E) : \mathbb{Q})]$ .
  4. Consider the group  $S_n$  of all permutations of the set  $\{1, 2, \dots, n\}$ .
    - I. Show that the transpositions  $(1 \ 2), (2 \ 3), \dots, (n-1 \ n)$  generate the whole group  $S_n$ .
    - II. Show that  $S_n$  is generated by the following two permutations:  $\rho = (1 \ 2 \ \dots \ n)$  and  $\sigma = (1 \ 2)$  (Hint: conjugate  $\sigma$  by  $\rho$ .)
    - III. For  $p$  is a prime,  $\rho$  a  $p$ -cycle, and  $\sigma$  a transposition, show that  $\rho$  and  $\sigma$  generate  $S_p$ . Show, by counterexample, that the hypothesis of  $p$  being prime cannot be removed.

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