Math 402 - 01 Homework (Spring 2019)

$\newcommand{\aut}{\textrm{Aut}} \newcommand{\inn}{\textrm{Inn}} \newcommand{\sub}{\textrm{Sub}} \newcommand{\cl}{\textrm{cl}} \newcommand{\join}{\vee} \newcommand{\bigjoin}{\bigvee} \newcommand{\meet}{\wedge} \newcommand{\bigmeet}{\bigwedge} \newcommand{\normaleq}{\unlhd} \newcommand{\normal}{\lhd} \newcommand{\union}{\cup} \newcommand{\intersection}{\cap} \newcommand{\bigunion}{\bigcup} \newcommand{\bigintersection}{\bigcap} \newcommand{\sq}[2][\ ]{\sqrt[#1]{#2\,}} \newcommand{\pbr}[1]{\langle #1\rangle} \newcommand{\ds}{\displaystyle} \newcommand{\C}{\mathbb{C}} \newcommand{\R}{\mathbb{R}} \newcommand{\Q}{\mathbb{Q}} \newcommand{\Z}{\mathbb{Z}} \newcommand{\N}{\mathbb{N}} \newcommand{\A}{\mathbb{A}} \newcommand{\F}{\mathbb{F}} \newcommand{\T}{\mathbb{T}} \newcommand{\ol}[1]{\overline{#1}} \newcommand{\imp}{\Rightarrow} \newcommand{\rimp}{\Leftarrow} \newcommand{\pinfty}{1/p^\infty} \newcommand{\power}{\mathcal{P}} \newcommand{\calL}{\mathcal{L}} \newcommand{\calC}{\mathcal{C}} \newcommand{\calN}{\mathcal{N}} \newcommand{\calB}{\mathcal{B}} \newcommand{\calF}{\mathcal{F}} \newcommand{\calR}{\mathcal{R}} \newcommand{\calS}{\mathcal{S}} \newcommand{\calU}{\mathcal{U}} \newcommand{\calT}{\mathcal{T}} \newcommand{\gal}{\textrm{Gal}} \newcommand{\isom}{\approx} \newcommand{\idl}{\textrm{Idl}} \newcommand{\lub}{\textrm{lub}} \newcommand{\glb}{\textrm{glb}} \newcommand{\cis}{\textrm{cis}} $

Problem Set 10 (complete) Due: 05/10/2019

1. Let F be a field of characteristic zero, $a\in F$, and $\xi=\xi_n$ a primitive n-th root of unity.
   1. Show by example that $\gal_F(F(\xi))$ need not be all of U_n.
   2. Show by example that $\gal_{F(\xi)}(F(\xi,\sq[n]{a}))$ need not be all of C_n.
2. Let G and H be solvable groups. Prove that $G\times H$ is solvable.
3. Show that the change of variable $y=x+(a/3)$ transforms the general cubic equation \[x^3+ax^2+bx+c = 0 \] into a depressed cubic. Therefore, Cardano's formula is useful to solve any cubic equation.

Problem Set 09 (complete) Due: 05/03/2019 Board presentation: 05/10/2019

1. Prove that the homomorphism \[\begin{array}{rccc} \psi:& U_n &\to &\gal(\Q(\xi_n)/\Q) \\ & k &\mapsto &\psi_k \\ \end{array}\] is surjective and injective.
2. Let $\xi_{15}=\cis(2\pi/15)$ be a primitive 15-th root of unity.
   1. Find the group $\gal(\Q(\xi_{15})/\Q)$ and draw its lattice of subgroups.
   2. Find and draw the lattice of intermediate fields of the extension $\Q(\xi_{15})/\Q$.
   3. Write down the correspondence between the subgroups in part 1, and the subfields in part 2, using the Fundamental Theorem of Galois Theory.
3. Show that any non-abelian simple group is non-solvable.
4. Show that if d is a divisor of n then $\Q(\xi_d)$ is a subfield of $\Q(\xi_n)$. Conclude that

$\varphi(d)$ divides $\varphi(n)$, and U_d is a quotient of U_n.

Problem Set 08 (complete) Due: 04/26/2019 Board presentation: 05/03/2019

1. Prove the following corollary to the Fundamental Theorem of Galois Theory. Use only the FTGT statements to prove it. Let E/F be a (finite) Galois extension, with Galois group $G=\gal_F(E)$. Let $L_1,L_2\in\sub_F(E)$ and $H_1,H_2\in\sub(G)$.
    1. $(L_1\meet L_2)^* = L_1^* \join L_2^*$
    2. $(L_1\join L_2)^* = L_1^* \meet L_2^*$
    3. $(H_1\meet H_2)^* = H_1^* \join H_2^*$
    4. $(H_1\join H_2)^* = H_1^* \meet H_2^*$
2. Let $f(x)\in\Q[x]$ be such that it has a non-real root. Let E be the splitting field of $f(x)$ over $\Q$. Prove that $\gal_\Q(E)$ has even order.
3. Consider the polynomial $f(x)=x^3+2x^2+2x+2\in\Q[x]$, and E its splitting field over $\Q$.
    1. Show that $f(x)$ has exactly one real root. (Hint: use calculus)
    2. Show that $f(x)$ is irreducible over $\Q$.
    3. Find $[E:\Q]$. Fully explain your calculation.
    4. Determine $\gal_\Q(E)$.
4. Consider the group S_n of all permutations of the set $\{1,2,\dots,n\}$.
    1. Show that the transpositions $(1\ \ 2),(2\ \ 3),\dots,(n-1\ \ n)$ generate the whole group S_n.
    2. Show that S_n is generated by the following two permutations: \[\rho = (1\ \ 2\ \ \dots\ \ n) \quad \text{and} \quad \sigma=(1\ \ 2) \] (Hint: conjugate σ by ρ.)
    3. For p is a prime, ρ a p-cycle, and σ a transposition, show that ρ and σ generate S_p. Show, by counterexample, that the hypothesis of p being prime cannot be removed.

Previous Homework

Home