Math 401 - 01 Daily Topics - part 3 (Fall 2018)

$\newcommand{\aut}{\textrm{Aut}} \newcommand{\inn}{\textrm{Inn}} \newcommand{\sub}{\textrm{Sub}} \newcommand{\cl}{\textrm{cl}} \newcommand{\join}{\vee} \newcommand{\bigjoin}{\bigvee} \newcommand{\meet}{\wedge} \newcommand{\bigmeet}{\bigwedge} \newcommand{\normaleq}{\unlhd} \newcommand{\normal}{\lhd} \newcommand{\union}{\cup} \newcommand{\intersection}{\cap} \newcommand{\bigunion}{\bigcup} \newcommand{\bigintersection}{\bigcap} \newcommand{\sq}[2][\ ]{\sqrt[#1]{#2\,}} \newcommand{\pbr}[1]{\langle #1\rangle} \newcommand{\ds}{\displaystyle} \newcommand{\C}{\mathbb{C}} \newcommand{\R}{\mathbb{R}} \newcommand{\Q}{\mathbb{Q}} \newcommand{\Z}{\mathbb{Z}} \newcommand{\N}{\mathbb{N}} \newcommand{\A}{\mathbb{A}} \newcommand{\F}{\mathbb{F}} \newcommand{\T}{\mathbb{T}} \newcommand{\ol}[1]{\overline{#1}} \newcommand{\imp}{\Rightarrow} \newcommand{\rimp}{\Leftarrow} \newcommand{\pinfty}{1/p^\infty} \newcommand{\power}{\mathcal{P}} \newcommand{\calL}{\mathcal{L}} \newcommand{\calC}{\mathcal{C}} \newcommand{\calN}{\mathcal{N}} \newcommand{\calB}{\mathcal{B}} \newcommand{\calF}{\mathcal{F}} \newcommand{\calR}{\mathcal{R}} \newcommand{\calS}{\mathcal{S}} \newcommand{\calU}{\mathcal{U}} \newcommand{\calT}{\mathcal{T}} \newcommand{\gal}{\textrm{Gal}} \newcommand{\isom}{\approx} \newcommand{\idl}{\textrm{Idl}} \newcommand{\lub}{\textrm{lub}} \newcommand{\glb}{\textrm{glb}} $

Home

Week 12	Topics
11/05/2018	Sylow Theorems
	Examples: (1) $\|G\|=35$ (2) $\|G\|=455$ (3) $\|G\|=21$ (4) $\|G\|=256$
11/06/2018	Test 2
11/07/2018	Rings. Definitions: ring, unity, ring with unity (unitary ring), commutative ring, units of a unitary ring
	Examples
	Prop: The units of a ring, $U(R)$ form a multiplicative group.
11/09/2018	No class.

Week 13	Topics
11/12/2018	Thm. 12.1
	Thm. 12.2
	Subrings, definition, examples
	Direct Products (Sums), definition, examples
	Ring homomorphisms, definition
	kernel, Ideal
	Homo, mono, epi, iso, endo, auto
11/13/2018	Test 2 returned
	R/I definition
	Thm. 12.3
	Integral Domains, zero-divisors
	Prop. Let R be a commutative ring. TFAE
	(1) R has no zero-divisors
	(2) R satisfies the cancellation law: $ab=ac$ and $a\neq 0 \imp b=c$.
	(3) R satisfies: $ab=0 \imp a=0\ \text{or}\ b=0$

	Definition: integral domain
	Examples: $\Z$, $\Q$, $\R$, $\C$, $\Q(\sqrt{2})$, $\Z_p$.
	Thm. (1) Any field is an integral domain.
	(2) Any finite ID is a field.
	Cor: $\Z_n$ is a field iff it is an ID iff n is a prime.
11/14/2018	Examples: $\Q(\sqrt{2})$ is a field.
	$\Z_3[i]$ is a field.
	$\Z_5[i]$ is not a field.
	Prop: If R is an ID, then $R[x]$ is an ID, and for any $f,g\in R[x]$ we have $\deg(fg)=\deg(f)+\deg(g)$.
	Example: $\Z_6[x]$ is not an ID and the degree formula does not hold.
11/16/2018	Snow day. Class cancelled.

Week 14	Topics
11/19/2018	
	(1) $\pbr{a}:=aR=\{ar\|r\in R\}$ is an ideal of R.
	(2) $a\in\pbr{a}$.
	(3) If $I\normaleq R$ and $a\in I$ then $\pbr{a} \leq I$.
	Def: $\pbr{a}$ is called the ideal generated by a. It is the smallest ideal of R that contains a.
	Example: In the ring $\Q[x]/\pbr{x^2-2}$ the element $u=x+I$ where $I=\pbr{x^2-2}$, satisfies $u^2=2$, i.e. it is a root of the polynomial x^2-2.
	Characteristic of a ring. Thms. 13.3 and 13.4.
11/20/2018	Comparison of $\Q[\sqrt{2}]$ and $\Q[x]/\pbr{x^2-2}$. Intuitive motivation for the construction $\Q[x]/\pbr{x^2-2}$.
	Given a commutative ring with unity R, and an ideal $I\normaleq R$,
	(Q1) when is R/I an I.D.?
	(Q2) when is R/I a field?
	Def: prime ideal
	Thm 14.3. R/I is an ID iff I is a prime ideal.

Week 15	Topics
11/26/2018	Lemma: Let R be a commutative ring with unity, and $I, J\normaleq R$.
	(1) * $I\intersection J \normaleq R$
	* $I\intersection J \leq I, J$
	* if $K\normaleq R$ and $K\leq I, J$ then $K\leq I\intersection J$.
	(2) * $I+J := \{x+y\|x\in I, y\in J\} \normaleq R$
	* $I, J \leq I+J$
	* if $K\normaleq R$ and $I, J\leq K$ then $I+J\leq K$.
	Prop: The set $\idl(R):=\{I\|I\normaleq R\}$ of ideals of R is a lattice, i.e. a partially ordered set, in which any two elements have a $\glb$ and a $\lub$.
	Cor: Let R be a commutative ring with unity and $I\normaleq R$. If I is maximal then it is prime.
11/27/2018	Board presentation, PS 11.
	Example: $R=\Z[x]$, $I=\pbr{x}$ is a prime ideal but it is not maximal.
	Fact: In the ring $\Z$ every ideal is a principal ideal, and every prime ideal is maximal.
11/28/2018	Def: Principal ideal domain (PID).
	Prop: $\Z$ is a principal ideal domain.
	Thm: If R is a PID and $I\normaleq R$ is prime then I is a maximal proper ideal.
	Cor: $\Z[x]$ is not a PID.
	Example: in $\Z[x]$ the ideal $K=\pbr{2}+\pbr{x}$ is not a principal ideal.
	Chapter 15. Divisibility by 9 criterion.

	Divisibility by 7 criterion: $n=10m+d_0$ is divisible by 7 iff $m-2d_0$ is divisible by 7
11/30/2018	Thm 15.1
	Thm. 15.3
	Lemma: Let R be a ring with unity. For $a,b\in R$, $n,m\in \Z$, $(m\cdot a)(n\cdot b)=(mn)\cdot(ab)$
	Thm. 15.5
Week 16	**Topics**
12/03/2018	Corollaries 1, 2, and 3.
	Field of fractions(quotients)
12/04/2018	Thm. 15.6 Moreover, F is minimal. If E is a field that contains a copy of D, then E contains a copy of F.
	Examples. 1) The field of fractions of $\Z$ is $\Q$.
	2) Let D be an integral domain, and $D[x]$ the ring of polynomials over D. The field of fractions of $D[x]$ is denoted by $D(x)$, and its elements are called *rational functions* over D. A *rational function* is a quotient of two polynomials $f(x)/g(x)$, with $g(x)\neq 0$.
	External and internal direct product of groups.
	Def: Internal semi direct product of groups. Given a group G, $N\normaleq G$, $H\leq G$ such that $N\intersection H=1$ and $NH=G$, we say that G is the (internal) semi direct product of N and H.
12/05/2018	Def: External semi direct product. Given two groups N and H and a homomorphism $\alpha:H\to\aut(N)$, write $\alpha(h)$ as α_h. Consider the cartesian product $N\times H$ with the following operation: $$(n_1,h_1)(n_2,h_2)=(n_1\alpha_{h_1}(n_2),h_1h_2)$$
	Thm: 1) The operation just defined makes $N\times H$ into a group. We denote it by $N\rtimes_\alpha H$. We omit the subscript α is it is understood from the context.
	2) $\bar{N}=\{(n,1)\mid n\in N\}$ is a normal subgroup of $N\rtimes_\alpha H$, isomorphic to N, via the map $$N\to N\rtimes_\alpha H\\ n\mapsto \bar{n}=(n,1)$$.
	3) $\bar{H}=\{(1,h)\mid h\in H\}$ is a subgroup of $N\rtimes_\alpha H$, isomorphic to H, via the map $$H\to N\rtimes_\alpha H\\ h\mapsto \bar{h}=(1,h)$$.
	4) $\bar{N}\intersection \bar{H}=1$ and $\bar{N}\bar{H}=N\rtimes_\alpha H$.
	5) $N\rtimes_alpha H$ is the internal semi direct product of $\bar{N}$ and $\bar{H}$.
	6) Given $h\in H$ and $n\in N$, conjugation of $\bar{n}$ by $\bar{h}$ is given by $$\varphi_{\bar{h}}(\bar{n}) =\overline{\alpha_h(n)}.$$
	Cor: When α is the trivial homomorphism, i.e. $\alpha_h=1$ for all $h\in H$, then the semi direct product is equal to the direct product, $N\rtimes_\alpha H=N\oplus H$.
	Cor: The operation in $N\rtimes_\alpha H$ is completely determined by the operations in N and H, and the relation $$\bar{h}\ \bar{n}=\overline{\alpha_h(n)}\ \bar{h}.$$
	Example: Let $N=\pbr{a}$ be cyclic of order 7, and $H=\pbr{b}$ cyclic of order 3. $\aut(N)\isom U_7$ is abelian of order 6, hence cyclic.
	$s:N\to N, a\mapsto a^2$ is an automorphism of N of order 3 since $a^{2^3}=a^8=a$.
	$\aut(N)$ is generated by $c:N\to N, a\mapsto a^3$, and $s=c^2$, since $a^{3^2}=a^9=a^2$.
	Any homomorphism $\alpha:H\to\aut(N)$ has to map b, which has order 3, to an element of $\aut(N)$ of order a divisor of 3. The only such elements are 1, s and $s^{-1}=c^4=s^2$.
12/07/2018	Board presentation PS 12
	Continuation of example. There are three different semi direct product of N and H, given by the three automorphisms $\alpha(b)=1$, $\beta(b)=s$, and $\gamma(b)=s^2$. Let's write down the three.
	Case 1: $\alpha(b)=1$ is trivial. In this case $N\rtimes_\alpha H = N\oplus H\isom C_{21}$.
	Case 2: $\beta(b)=s$. $$ba = \beta_b(a)b = s(a)b = a^2b$$
	so $N\rtimes_\beta H$ is not abelian, and not isomorphic to case 1.
	Case 3: To distinguish from case 2, let's write $N=\pbr{u}$ cyclic of order 7, and $H=\pbr{v}$ cyclic of order 3, $\gamma(v)=s^2$. $$vu = \gamma_v(u)v = s^2(u)v = u^4v$$
	Again $N\rtimes_\gamma H$ is not abelian, not isomorphic to case 1.
	Claim: $N\rtimes_\beta H$ is isomorphic to $N\rtimes_\gamma H$ via the map $a\mapsto u, b\mapsto v^{-1}$.
	Cor: there are only two non-isomorphic semi direct products of a cyclic group of order 7 and a cyclic group of order 3, namely, the direct product, and the non-abelian semi direct product of case 2.

	Example: Let G be a group of order 21. By Sylow's theorem we have $n_7=1$. Let N be the Sylow 7-subgroup of G. We also know that n_3 is either 1 or 7. Let H be a Sylow 3-subgroup of G. When $n_3=1$ H is a normal subgroup of G and G is the direct product of N and H. When $n_3=7$, then H is not normal, and G is the non-abelian semi direct product of N and H.
	Therefore, there are exactly two non-isomorphic groups of order 21.

Daily topics (2)

Home