The McKay–Navarro Conjecture: The Conjecture That Keeps on Giving!

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The Basic Set-Up

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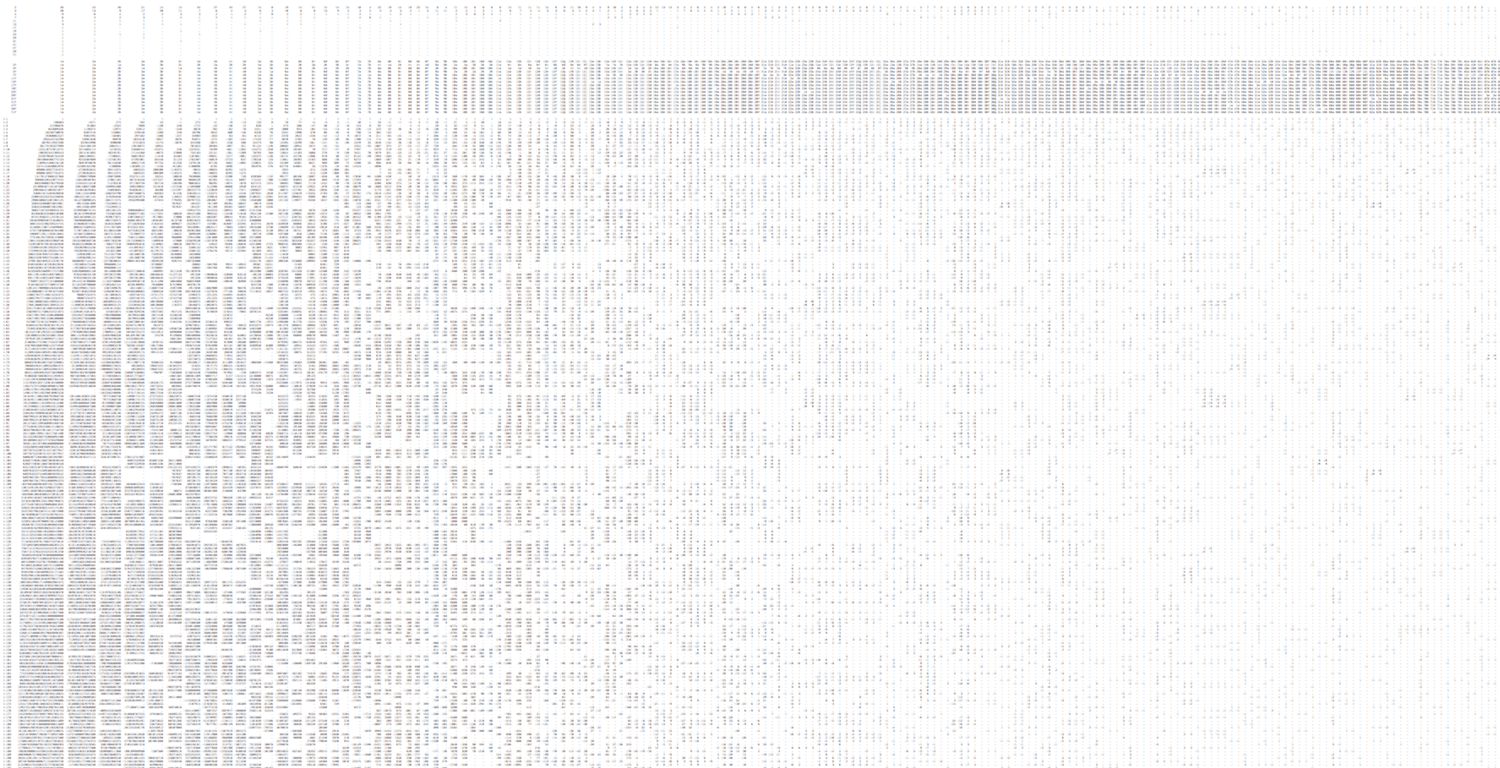
    - entries: $\chi(g)$

Example: The 194 × 194 Character Table of the “Monster"



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2. In general, what can we say about the group from the character table? (A classical question!)

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- The “local-global" conjectures relate the representation theory of G to that of certain proper subgroups
    - e.g. normalizer $N_G(P)$ of a Sylow subgroup.
- Often, these have to do with the number of characters of the group of a given type.

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Fix a prime p and a (Sylow) p-subgroup P of G

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Local Side		Global Side
Say something about $N_G(P)$ or $\mathrm{Irr}(N_G(P))$	$\longleftrightarrow$	Say something about $\mathrm{Irr}(G)$

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Write $|\mathrm{Irr}_{p'}(G)|$ for the number of $\chi \in \mathrm{Irr}(G)$ with $p \nmid \chi(1)$

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Conjecture

Let G be a finite group, p a prime, and $P \in \mathrm{Syl}_p(G)$. Then

$\mathrm{Irr}_{p'}(G) \longleftrightarrow \mathrm{Irr}_{p'}(N_G(P))$

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(N_G(P))|.$$

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- But for p odd, still open (lots of ongoing work...)
- Lots of refinements!

Ex: Character Table of Sylow 71-Normalizer of Monster

X.1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X.2	1	1	1	1	1	1	1	1	1	E	E	E	E	E	E	E	F	F	F	F	F	F	F	/E	/E	/E	/E	/E	/E	/E	/F	/F	/F	/F	/F	/F
X.3	1	1	1	1	1	1	1	1	1	F	F	F	F	F	F	F	/E	/E	/E	/E	/E	/E	/E	/F	/F	/F	/F	/F	/F	/F	E	E	E	E	E	E
X.4	1	1	1	1	1	1	1	1	1	/F	/F	/F	/F	/F	/F	/F	E	E	E	E	E	E	E	F	F	F	F	F	F	F	/E	/E	/E	/E	/E	/E
X.5	1	1	1	1	1	1	1	1	1	/E	/E	/E	/E	/E	/E	/E	/F	/F	/F	/F	/F	/F	/F	E	E	E	E	E	E	E	F	F	F	F	F	F
X.6	1	1	1	B	D	C	/B	/D	/C	D	/D	/B	/C	C	1	B	/B	1	C	B	/C	D	/D	/C	/B	B	1	/D	C	D	C	D	/C	/D	B	/B
X.7	1	1	1	C	/B	/D	/C	B	D	/B	B	/C	D	/D	1	C	/C	1	/D	C	D	/B	B	D	/C	C	1	B	/D	/B	/D	/B	D	B	C	/C
X.8	1	1	1	D	C	/B	/D	/C	B	C	/C	/D	B	/B	1	D	/D	1	/B	D	B	C	/C	B	/D	D	1	/C	/B	C	/B	C	B	/C	D	/D
X.9	1	1	1	/D	/C	B	D	C	/B	/C	C	D	/B	B	1	/D	D	1	B	/D	/B	/C	C	/B	D	/D	1	C	B	/C	B	/C	/B	C	/D	D
X.10	1	1	1	/C	B	D	C	/B	/D	B	/B	C	/D	D	1	/C	C	1	D	/C	/D	B	/B	/D	C	/C	1	/B	D	B	D	B	/D	/B	/C	C
X.11	1	1	1	/B	/D	/C	B	D	C	/D	D	B	C	/C	1	/B	B	1	/C	/B	C	/D	D	C	B	/B	1	D	/C	/D	/C	/D	C	D	/B	B
X.12	1	1	1	B	D	C	/B	/D	/C	G	/J	M	/Q	R	/F	/L	K	E	P	/N	/O	I	/H	/R	L	/M	F	/G	Q	J	O	H	/P	/I	/K	N
X.13	1	1	1	B	D	C	/B	/D	/C	H	/I	N	/P	O	/E	/K	M	/F	R	/L	/Q	G	/J	/O	K	/N	E	/H	P	I	Q	J	/R	/G	/M	L
X.14	1	1	1	B	D	C	/B	/D	/C	I	/H	K	/O	P	E	/N	L	F	Q	/M	/R	J	/G	/P	N	/K	/E	/I	O	H	R	G	/Q	/J	/L	M
X.15	1	1	1	B	D	C	/B	/D	/C	J	/G	L	/R	Q	F	/M	N	/E	O	/K	/P	H	/I	/Q	M	/L	/F	/J	R	G	P	I	/O	/H	/N	K
X.16	1	1	1	C	/B	/D	/C	B	D	K	/N	/O	I	/H	E	P	/R	F	/G	Q	J	L	/M	H	/P	O	/E	/K	/I	N	/J	M	G	/L	R	/Q
X.17	1	1	1	C	/B	/D	/C	B	D	L	/M	/R	J	/G	F	Q	/P	/E	/I	O	H	N	/K	G	/Q	R	/F	/L	/J	M	/H	K	I	/N	P	/O
X.18	1	1	1	C	/B	/D	/C	B	D	M	/L	/Q	G	/J	/F	R	/O	E	/H	P	I	K	/N	J	/R	Q	F	/M	/G	L	/I	N	H	/K	O	/P
X.19	1	1	1	C	/B	/D	/C	B	D	N	/K	/P	H	/I	/E	O	/Q	/F	/J	R	G	M	/L	I	/O	P	E	/N	/H	K	/G	L	J	/M	Q	/R
X.20	1	1	1	D	C	/B	/D	/C	B	O	/P	/I	/K	N	/E	H	/J	/F	M	G	/L	R	/Q	/N	/H	I	E	/O	K	P	L	Q	/M	/R	J	/G
X.21	1	1	1	D	C	/B	/D	/C	B	P	/O	/H	/N	K	E	I	/G	F	L	J	/M	Q	/R	/K	/I	H	/E	/P	N	O	M	R	/L	/Q	G	/J
X.22	1	1	1	D	C	/B	/D	/C	B	Q	/R	/G	/M	L	F	J	/I	/E	N	H	/K	O	/P	/L	/J	G	/F	/Q	M	R	K	P	/N	/O	I	/H
X.23	1	1	1	D	C	/B	/D	/C	B	R	/Q	/J	/L	M	/F	G	/H	E	K	I	/N	P	/O	/M	/G	J	F	/R	L	Q	N	O	/K	/P	H	/I
X.24	1	1	1	/D	/C	B	D	C	/B	/R	Q	J	L	/M	F	/G	H	/E	/K	/I	N	/P	O	M	G	/J	/F	R	/L	/Q	/N	/O	K	P	/H	I
X.25	1	1	1	/D	/C	B	D	C	/B	/Q	R	G	M	/L	/F	/J	I	E	/N	/H	K	/O	P	L	J	/G	F	Q	/M	/R	/K	/P	N	O	/I	H
X.26	1	1	1	/D	/C	B	D	C	/B	/P	O	H	N	/K	/E	/I	G	/F	/L	/J	M	/Q	R	K	I	/H	E	P	/N	/O	/M	/R	L	Q	/G	J
X.27	1	1	1	/D	/C	B	D	C	/B	/O	P	I	K	/N	E	/H	J	F	/M	/G	L	/R	Q	N	H	/I	/E	O	/K	/P	/L	/Q	M	R	/J	G
X.28	1	1	1	/C	B	D	C	/B	/D	/N	K	P	/H	I	E	/O	Q	F	J	/R	/G	/M	L	/I	O	/P	/E	N	H	/K	G	/L	/J	M	/Q	R
X.29	1	1	1	/C	B	D	C	/B	/D	/M	L	Q	/G	J	F	/R	O	/E	H	/P	/I	/K	N	/J	R	/Q	/F	M	G	/L	I	/N	/H	K	/O	P
X.30	1	1	1	/C	B	D	C	/B	/D	/L	M	R	/J	G	/F	/Q	P	E	I	/O	/H	/N	K	/G	Q	/R	F	L	J	/M	H	/K	/I	N	/P	O
X.31	1	1	1	/C	B	D	C	/B	/D	/K	N	O	/I	H	/E	/P	R	/F	G	/Q	/J	/L	M	/H	P	/O	E	K	I	/N	J	/M	/G	L	/R	Q
X.32	1	1	1	/B	/D	/C	B	D	C	/J	G	/L	R	/Q	/F	M	/N	E	/O	K	P	/H	I	Q	/M	L	F	J	/R	/G	/P	/I	O	H	N	/K
X.33	1	1	1	/B	/D	/C	B	D	C	/I	H	/K	O	/P	/E	N	/L	/F	/Q	M	R	/J	G	P	/N	K	E	I	/O	/H	/R	/G	Q	J	L	/M
X.34	1	1	1	/B	/D	/C	B	D	C	/H	I	/N	P	/O	E	K	/M	F	/R	L	Q	/G	J	O	/K	N	/E	H	/P	/I	/Q	/J	R	G	M	/L
X.35	1	1	1	/B	/D	/C	B	D	C	/G	J	/M	Q	/R	F	L	/K	/E	/P	N	O	/I	H	R	/L	M	/F	G	/Q	/J	/O	/H	P	I	K	/N
X.36	35	A	/A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
X.37	35	/A	A	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

Ex: Characters of the “Monster" with Degree Prime to 71

X.1	1	1	1	1	1	1	1	1	1	1	1	1
X.4	842609326	1139374	12974	55912	-221	-248	8878	782	-82	78	2451	-49
X.8	3879214937598	337044990	-690690	4751823	-4173	-3876	345598	-3874	-258	286	54473	98
X.12	222879856734249	3864186921	-7196631	26057022	11310	34503	1187881	-39831	1577	1001	132374	-126
X.16	8980616927734375	-2720265625	39414375	1603525	200200	-116375	-9625	-20825	8295	1575	.	.
X.17	8980616927734375	-2720265625	39414375	1603525	200200	-116375	-9625	-20825	8295	1575	.	.
X.26	3503434660075044981	-89143381899	-755269515	.	.	767637	184437	161109	39285	18837	-8019	4356
X.27	3503434660075044981	-89143381899	-755269515	.	.	767637	184437	161109	39285	18837	-8019	4356
X.39	172399434201593354756	1932029655556	531831300	-441097657	-6179173	-1560013	2288132	-476476	-9212	-22204	-61369	1631
X.40	172399434201593354756	1932029655556	531831300	-441097657	-6179173	-1560013	2288132	-476476	-9212	-22204	-61369	1631
X.41	286243267692724486144	-53936390144	7514357760	-281380736	7326592	1646008	1835008	.	.	.	-30856	11144
X.42	286243267692724486144	-53936390144	7514357760	-281380736	7326592	1646008	1835008	.	.	.	-30856	11144
X.44	640558364167263622626	-1925003475486	9988806114	.	-3720087	.	26082	-584766	9954	-18942	39501	13251
X.45	640558364167263622626	-1925003475486	9988806114	.	-3720087	.	26082	-584766	9954	-18942	39501	13251
X.49	776097192277137500000	29144516700000	11163740000	10339920710	6141590	5084000	66037600	106080	-20640	36960	1312500	.
X.53	1480279477146615234375	90807234375	-1811005625	-593746725	-4751175	-5742750	2498375	-251225	154695	-64025	.	.
X.54	1480279477146615234375	90807234375	-1811005625	-593746725	-4751175	-5742750	2498375	-251225	154695	-64025	.	.
X.81	149614794149226010902528	.	-165356240896	-3731718144	20917248	-22165248	.	.	.	.	374528	2528
X.82	149614794149226010902528	.	-165356240896	-3731718144	20917248	-22165248	.	.	.	.	374528	2528
X.91	261575621299360905468750	321725078268750	-83502851250	-1950194610	32014710	23122125	64355150	7794670	-475570	-246610	-1312500	.
X.101	600020772685064502392907	1967052076247115	328187590731	178514751987	.	.	320575563	13586859	-203445	-3861	558657	8532
X.102	626877403613887304040448	.	.	-6897532928	-65601536	26411008	.	.	.	.	360448	40448
X.103	626877403613887304040448	.	.	-6897532928	-65601536	26411008	.	.	.	.	360448	40448
X.105	689763222744895005949242	-389556578898630	-206943847110	.	.	767637	50535738	8055450	707130	-263718	-1058508	30492
X.106	689763222744895005949242	-389556578898630	-206943847110	.	.	767637	50535738	8055450	707130	-263718	-1058508	30492
X.123	5514132424881463208443904	2642676197359616	.	65393917952	47151104	30507008	.	33554432	.	.	-1284096	43904
X.124	5514132424881463208443904	-2642676197359616	.	65393917952	47151104	30507008	.	.	.	.	-1284096	43904
X.125	5514132424881463208443904	-2642676197359616	.	65393917952	47151104	30507008	.	.	.	.	-1284096	43904
X.128	7567151576542452425781250	-3099399926218750	511362101250	89656289800	-44244200	28861000	-85450750	-16285150	-680190	122850	.	.
X.129	7567151576542452425781250	-3099399926218750	511362101250	89656289800	-44244200	28861000	-85450750	-16285150	-680190	122850	.	.
X.133	9416031858681585751556096	9211433539600384	.	358364217344	-97936384	-26411008	.	16777216	.	.	-1915904	-43904
X.135	9592298143650890255171584	-1807462370115584	111379480576	6471756928	-253425536	51026248	42205184	.	.	.	-339416	-3416
X.136	9592298143650890255171584	-1807462370115584	111379480576	6471756928	-253425536	51026248	42205184	.	.	.	-339416	-3416
X.149	38471795739256565080575180	11754324719717580	-2421327344436	90098990982	-40783743	-30390633	-335774516	5501132	562380	562380	1075305	-12320
X.152	42001454087954515167503490	8436234038236290	1759984379010	178514751987	.	.	-321866622	12478050	750978	55458	1015740	6615
X.175	129572518017902934396764160	11854109736960000	.	-77108871168	160183296	56918016	.	-16777216	.	.	-291840	-35840
X.177	135226984222789977095703125	8295452743703125	-4497154536875	-318660767425	37948625	46899125	335383125	13265525	467285	847925	.	.

A More Managable Example...

Example (Character Table of $G = A_5$)

Class → Character↓	(1)	(12)(34)	(123)	(12345)	(13524)
χ_1	1	1	1	1	1
χ_2	3	-1	0	α	α'
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χ_5	5	1	-1	0	0

$\alpha = 1 + \zeta_5 + \zeta_5^4$
$\alpha' = 1 + \zeta_5^2 + \zeta_5^3$ where $\zeta_5 := e^{2\pi i/5}$

Case $p = 2$

Example ($\mathrm{Irr}_{2'}(A_5)$:)

Class → Character↓	(1)	(12)(34)	(123)	(12345)	(13524)
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Character Table of $N_G(P)$:

χ_1	1	1	1	1
χ_2	1	1	β	$\overline{\beta}$
χ_3	1	1	$\overline{\beta}$	β
χ_4	3	-1	0	0

$$\beta = \zeta_3$$

Case $p = 3$

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- These iMN conditions expand on the inductive McKay conditions of Isaacs–Malle–Navarro

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(2) extensions to automorphism groups of $\chi, \Omega(\chi)$ must "behave the same" with respect to $\mathcal{H}$

$\mathrm{Aut}(G)_{P,\chi}$ $\tilde{\chi}$ $\tilde{\chi}^\sigma = \tilde{\chi}\beta_1$

$\tilde{\Omega}(\chi)$ $\tilde{\Omega}(\chi)^\sigma = \tilde{\Omega}(\chi)\beta_2$

$\chi^\sigma = \chi$

$\Rightarrow \beta_1 = \beta_2$

$\Omega(\chi)^\sigma = \Omega(\chi)$

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  - in particular, iMN now complete for $p = 2$ and G of type $B, C, G_2, F_4, E_7, E_8, {}^3D_4$

Now to Reasonable Consideration 2: What Does $CT(G)$ Tell Us?

Recall: $CT(G)$ does *not* determine G!

Example

$$D_8 = \langle a, b \colon a^4 = b^2 = 1, bab = a^{-1} \rangle$$

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k : i^2 = j^2 = k^2 = -1\}$$

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$$D_8 \not\cong Q_8$$

Both have character table:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 \\ 2 & 0 & -2 & 0 & 0 \end{bmatrix}$$

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These are all what we’d call "global" properties

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Question (Brauer's Problem 12)

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Problem 12. Given the character table of a group G *and a prime* p *dividing* $n = |G|$, *how much information about the structure of the* p-*Sylow group* P *can be obtained? In particular, can it be decided whether or not* P *is Abelian?*

McKay–Navarro: a fountain of answers to Brauer's Problem 12

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$N_G(P) = P \iff \exists! \chi \in \mathrm{Irr}_{p'}(G)$ *satisfying* $\chi(g) \in \mathbb{Q}(e^{2\pi i/m})$ *for some* $(m, p) = 1$

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- $p = 2$ finished by Malle 2020

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- The **Principal Block** B_0 is the block with $1_G \in \mathrm{Irr}(B_0)$.

Example (2-Blocks of A_5: B_0, B_1)

Class → / Character↓	(1)	(12)(34)	(123)	(12345)	(13524)
χ_1	1	1	1	1	1
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Principal Block Version of McKay–Navarro $\rightsquigarrow$ More Answers to Brauer Problem 12

Normal p-Complements

p odd:

Theorem (Navarro-Tiep-Vallejo '19)

$N_G(P) = P \times V \iff \exists!\ \chi \in \mathrm{Irr}_{p'}(B_0)$ *satisfying* $\chi(g) \in \mathbb{Q}(e^{2\pi i/m})$ *for some* $(m, p) = 1$

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$p = 2$:

Theorem (Navarro–Vallejo ’17 + SF–Taylor ’18b)

$N_G(P) = P \times V \iff$ *every* $\chi \in \mathrm{Irr}_{2'}(B_0)$ *is fixed by the Galois Automorphism* σ_0.

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- $\sigma_1 \in \mathcal{G}$ fixes p'-roots of unity; $\sigma_1(\xi) = \xi^{p+1}$ if $|\xi|$ is a power of p

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Let $p = 2$. P is (minimally) generated by two nontrivial elements $\Leftrightarrow |\mathrm{Irr}_{2',\sigma_1}(B_0)| = 4$.

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- (Both of these false for $p \geq 5$!)

Examples with A_5

Example (2-Blocks of A_5: B_0, B_1)

Class → / Character↓	(1)	(12)(34)	(123)	(12345)	(13524)
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$\alpha = 1 + \zeta_5 + \zeta_5^4$; $\alpha' = 1 + \zeta_5^2 + \zeta_5^3$ with $\zeta_5 := e^{2\pi i/5}$

P is 2-generated, $\exists$ 4 σ_1-fixed characters in $\mathrm{Irr}_{2'}(B_0)$!

Example (3-Blocks of A_5: B_0, B_1, B_2)

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$\alpha = 1 + \zeta_5 + \zeta_5^4$; $\alpha' = 1 + \zeta_5^2 + \zeta_5^3$ with $\zeta_5 := e^{2\pi i/5}$

P is cyclic, $\exists$ 3 σ_1-fixed characters in $\mathrm{Irr}_{3'}(B_0)$!

Example (5-Blocks of A_5: B_0, B_1)

Class → Character↓	(1)	(12)(34)	(123)	(12345)	(13524)
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χ_2	3	-1	0	α	α'
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$\alpha = 1 + \zeta_5 + \zeta_5^4$; $\alpha' = 1 + \zeta_5^2 + \zeta_5^3$ with $\zeta_5 := e^{2\pi i/5}$

P is cyclic, but $\exists$ 4 σ_1-fixed characters in $\mathrm{Irr}_{5'}(B_0)$!

Questions?

Thank You!