Problem 6. For positive real numbers a, b, c and a positive integer n define $d_n = 2\sqrt[n]{a} - \sqrt[n]{b} - \sqrt[n]{c}$. For which a, b, c the series $\sum_{n=1}^{\infty} d_n$ converges?

Solution. Let $0 < u < w$ be real numbers. The mean value theorem applied to the function $f(x) = \sqrt[n]{x}$ yields

$$\frac{\sqrt[n]{w} - \sqrt[n]{u}}{w - u} = f'(t) = \frac{1}{n\sqrt[n]{t^{n-1}}}$$

for some t between u and w, i.e. $u < t < w$. It follows that

$$\frac{w-u}{n\sqrt[n]{w^{n-1}}} \leq \sqrt[n]{w} - \sqrt[n]{u} \leq \frac{w-u}{n\sqrt[n]{u^{n-1}}}.$$

The lower bound implies that the series

$$\sum_{n=1}^{\infty} (\sqrt[n]{w} - \sqrt[n]{u})$$

diverges and the upper bound implies that the series

$$\sum_{n=1}^{\infty} (\sqrt[n]{w} - \sqrt[n]{u})^2$$

converges. Note that

$$d_n = 2\sqrt[n]{a} - 2\sqrt[n]{\sqrt{bc}} - \left(\sqrt[n]{\sqrt{b}} - \sqrt[n]{\sqrt{c}}\right)^2.$$

Since the series $\sum_{n=1}^{\infty} \left(\sqrt[n]{\sqrt{b}} - \sqrt[n]{\sqrt{c}}\right)^2$ converges, we see that $\sum_{n=1}^{\infty} d_n$ converges if and only if the series $\sum_{n=1}^{\infty} \left(\sqrt[n]{a} - \sqrt[n]{\sqrt{bc}}\right)$ converges. As we observed earlier, the last series diverges unless $a = \sqrt{bc}$. Thus the series $\sum_{n=1}^{\infty} d_n$ converges if and only if $a = \sqrt{bc}$.

Excercise. For positive real numbers a, b, c, d and a positive integer n define $d_n = \sqrt[n]{a} + \sqrt[n]{d} - \sqrt[n]{b} - \sqrt[n]{c}$. For which a, b, c, d the series $\sum_{n=1}^{\infty} d_n$ converges?