**Problem 6.** Prove that
$$\sum_{k=0}^{n}\binom{3n}{3k}=\frac{2}{3}\left(2^{3n-1}+(-1)^n\right).$$

**Solution.** Let $\omega=(-1+\sqrt{-3})/2$, then $1+\omega+\omega^2=0$ and $\omega^3=1$. By the binomial theorem, we have
$$(1+\omega)^{3n}=\sum_{k=0}^{3n}\binom{3n}{k}\omega^k=\sum_{k=0}^{n}\binom{3n}{3k}+\omega\sum_{k=0}^{n}\binom{3n}{3k+1}+\omega^2\sum_{k=0}^{n}\binom{3n}{3k+2} \tag{1}$$
and
$$(1+\omega^2)^{3n}=\sum_{k=0}^{3n}\binom{3n}{k}\omega^{2k}=\sum_{k=0}^{n}\binom{3n}{3k}+\omega^2\sum_{k=0}^{n}\binom{3n}{3k+1}+\omega\sum_{k=0}^{n}\binom{3n}{3k+2} \tag{2}$$
Note now that $1+\omega=-\omega^2$ so
$$(1+\omega)^{3n}=(-\omega^2)^{3n}=(-1)^{3n}\omega^{6n}=(-1)^n,$$
and similarly $(1+\omega^2)^{3n}=(-1)^n$. Using this and adding (1) and (2) we get
$$2(-1)^2=2\sum_{k=0}^{n}\binom{3n}{3k}+(\omega+\omega^2)\sum_{k=0}^{n}\binom{3n}{3k+1}+(\omega^2+\omega)\sum_{k=0}^{n}\binom{3n}{3k+2}=$$
$$3\sum_{k=0}^{n}\binom{3n}{3k}-\sum_{k=0}^{n}\binom{3n}{3k}-\sum_{k=0}^{n}\binom{3n}{3k+1}-\sum_{k=0}^{n}\binom{3n}{3k+2}=3\sum_{k=0}^{n}\binom{3n}{3k}-\sum_{k=0}^{3n}\binom{3n}{k}=3\sum_{k=0}^{n}\binom{3n}{3k}-2^{3n}$$
(we used the equalities $\omega+\omega^2=-1$ and $\sum_{k=0}^{3n}\binom{3n}{k}=2^{3n}$).Thus
$$\sum_{k=0}^{n}\binom{3n}{3k}=\frac{2(-1)^n+2^{3n}}{3}=\frac{2}{3}\left(2^{3n-1}+(-1)^n\right).$$

**Excercise.** Find and prove similar formulas for
$$\sum_{k\equiv i \mod 3}\binom{n}{k}$$
for any $n$ and any $i\in\{0,1,2\}$.