Problem 4. Let $n > 0$ be an odd integer. Prove that there exists a set $S = \{A_1, \ldots, A_{2n}\}$ of $2n$ distinct points in the plane which are not collinear and such that if $i + j \neq 2n + 1$ then the line A_iA_j contains a third point from S.

Solution. We start by reviewing some basic trigonometry. Recall that

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha+\beta}{2}\right)$$

and

$$\sin\alpha - \sin\beta = -2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right).$$

It follows that

$$-\cot\left(\frac{\alpha+\beta}{2}\right) = \frac{\sin\alpha - \sin\beta}{\cos\alpha - \cos\beta} = \frac{\cos\alpha}{\cos\alpha - \cos\beta}\tan\alpha + \frac{-\cos\beta}{\cos\alpha - \cos\beta}\tan\beta. \tag{1}$$

From (1) we get the following lemma.

Lemma. For any α, β such that $\cos\alpha \neq 0 \neq \cos\beta$ and $\cos\alpha \neq \cos\beta$ the points $(\tan\alpha, \sec\alpha)$, $(\tan\beta, \sec\beta)$, $(-\cot((\alpha+\beta)/2), 0)$ are collinear.

Indeed, the line through the points $(\tan\alpha, \sec\alpha)$, $(\tan\beta, \sec\beta)$ consists of all points of the form $t(\tan\alpha, \sec\alpha) + (1-t)(\tan\beta, \sec\beta)$ for some $t \in \mathbb{R}$. Taking $t = \cos\alpha/(\cos\alpha - \cos\beta)$ we get the point $(-\cot((\alpha+\beta)/2), 0)$ by (1).

We are now ready to solve the problem. Let $n > 1$ be an odd integer. Fix α which is not of the form $k\pi/2n$ for k an integer. Define

$$A_k = \left(\tan\left(\alpha + \frac{2(k-1)\pi}{n}\right), \sec\left(\alpha + \frac{2(k-1)\pi}{n}\right)\right)$$

for $k = 1, \ldots, n$, and

$$A_k = \left(-\cot\left(\alpha + \frac{2(2n-k)\pi}{n}\right), 0\right)$$

for $k = n+1, n+2, \ldots, 2n$. It is easy to see that $A_i \neq A_j$ for $i \neq j$. We will show that the set $S = \{A_1, \ldots, A_{2n}\}$ has the required property. For any integer m let $B_m = \left(-\cot\left(\alpha + \frac{m\pi}{n}\right), 0\right)$. Writing $m = nu + r$ with $1 \leq r \leq n$ we see that $B_m = A_{2n-r/2}$ if r is even and $B_m = A_{2n-(r+n)/2}$ if r is odd. We see that for any m we have $\{A_{n+1}, \ldots, A_n\} = \{B_m, B_{m+1}, \ldots, B_{m+n-1}\}$.

Note that the points $A_{n+1}, \ldots, A_{2n}$ are collinear. From our Lemma we see that if $1 \leq i < j \leq n$ then the points A_i, A_j, B_{i+j-2} are collinear. Since B_{i+j-2} is one of the points $A_{n+1}, \ldots, A_{2n}$, the line A_iA_j contains a third point in S. Suppose now that $1 \leq i \leq n < j \leq 2n$. We have $A_j = B_m$ for some $m \in \{i-1, i, \ldots, i+n-2\}$. Taking $k = m - i + 2$ we have $1 \leq k \leq n$ and the points A_k, A_i, B_m are collinear. Note that $k = i$ if and only if $m = 2i - 2$, which is equivalent to $j = 2n + 1 - i$. Thus if $i + j \neq 2n + 1$ then the line A_iA_j contains a third point A_k in S. This completes our proof that S has the required property.

Remark. The above solution may look mysterious. A conceptually simpler solution can be given using projective geometry. Consider a regular n-gon $A_1, \ldots, A_n$ in the projective plane (which is obtained by adding the line at infinity to the affine plane). Let $n = 2k + 1$. The line $A_{i+k}A_{i+k+1}$ intersects the line at infinity at a point B_i. Set $A_i = B_{2n+1-i}$ for $i = n+1, \ldots, 2n$. Every line A_iA_j with $1 \leq i < j \leq n$ is parallel to one of the sides of the n-gon, so it passes through one of the points B_s, $s = 1, \ldots, n$. A line A_iB_j is simply the line through A_i parallel to the side $A_{j+k}A_{j+k+1}$, so it passes through a second vertex of the n-gon unless $i = j$. This shows that $S = \{A_1, \ldots, A_{2n}\}$ has the required property.

Remark. The above problem is connected to the following well-known theorem.

Sylvester-Gallai Theorem. Given any finite set of $n \geq 2$ points on the plane, there exists a line passing through exactly two points in S.

Question. Is the problem true for n even?