**Problem 4.** A sequence $a_1, a_2, \ldots$ of real numbers has the following properties:
(i) $|a_1 + a_2 + \ldots + a_k| \le 1$ for every $k$;
(ii) $|a_k - a_{k-1}| \le 1/k$ for every $k \ge 2$.
Suppose that $|a_k| \ge \dfrac{c}{\sqrt{k}}$ for infinitely many $k$. Prove that $c \le \sqrt{2}$.

**Solution.** Let $n$ be a positive integer. Note that

$$|a_{n+i} - a_n| = |(a_{n+1} - a_n) + (a_{n+2} - a_{n+1}) + \ldots + (a_{n+i} - a_{n+i-1})|$$

$$\le |a_{n+1} - a_n| + |a_{n+2} - a_{n+1}| + \ldots + |a_{n+i} - a_{n+i-1}| \le \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+i}.$$

Adding these inequalities for $i = 1, 2, \ldots, M$ we get

$$\sum_{i=1}^{M} |a_{n+i} - a_n| \le \sum_{j=0}^{M-1} \frac{M-j}{n+1+j} \le \sum_{j=0}^{M-1} \frac{M-j}{n+1} = \frac{M(M+1)}{2(n+1)}. \qquad (1)$$

Similarly, we have

$$|a_{n-i} - a_n| \le \frac{1}{n} + \frac{1}{n-1} + \ldots + \frac{1}{n-i+1}.$$

Adding these inequalities for $i = 1, 2, \ldots, N$ (where $N < n$) we get

$$\sum_{i=1}^{N} |a_{n-i} - a_n| \le \sum_{j=0}^{N-1} \frac{N-j}{n-j} \le \sum_{j=0}^{N-1} \frac{N-j}{n-N+1} = \frac{N(N+1)}{2(n-N+1)}. \qquad (2)$$

It follows from (1) and (2) that

$$\left|\sum_{i=n-N}^{n+M} a_i - (M+N+1)a_n\right| = \left|\sum_{i=1}^{N}(a_{n-i} - a_n) + \sum_{i=1}^{M}(a_{n+i} - a_n)\right| \le \frac{M(M+1)}{2(n+1)} + \frac{N(N+1)}{2(n-N+1)}. \quad (3)$$

On the other hand,

$$\left|\sum_{i=n-N}^{n+M} a_i\right| = \left|\sum_{i=1}^{n+M} a_i - \sum_{i=1}^{n-N-1} a_i\right| \le \left|\sum_{i=1}^{n+M} a_i\right| + \left|\sum_{i=1}^{n-N-1} a_i\right| \le 1 + 1 = 2$$

(we used property (i)). Thus

$$\left|\sum_{i=n-N}^{n+M} a_i - (M+N+1)a_n\right| \ge (N+M+1)|a_n| - \left|\sum_{i=n-N}^{n+M} a_i\right| \ge (N+M+1)|a_n| - 2. \qquad (4)$$

Combining (3) and (4) we get

$$(N+M+1)|a_n| - 2 \le \frac{M(M+1)}{2(n+1)} + \frac{N(N+1)}{2(n-N+1)} \le \frac{M(M+1) + N(N+1)}{2(n-N+1)}.$$

Taking $M = N$, we get

$$|a_n| \le \frac{1}{2N+1}\left(2 + \frac{N(N+1)}{n-N+1}\right).$$

We choose now $N = \lfloor\sqrt{2n}\rfloor$, so $\sqrt{2n} - 1 < N \le \sqrt{2n}$. Thus

$$|a_n| \le \frac{1}{2\sqrt{2n}-1}\left(2 + \frac{\sqrt{2n}(\sqrt{2n}+1)}{n-\sqrt{2n}}\right) = \frac{1}{2\sqrt{2n}-1}\left(2 + \frac{2(\sqrt{2n}+1)}{\sqrt{2n}-2}\right) = \frac{2}{\sqrt{2n}-2} = \frac{\sqrt{2}}{\sqrt{n}-\sqrt{2}}.$$

Suppose now that $|a_n| \ge c/\sqrt{n}$. Then

$$\frac{\sqrt{2}}{\sqrt{n}-\sqrt{2}} \ge \frac{c}{\sqrt{n}}$$

i.e.

$$c \leq \frac{\sqrt{2}\sqrt{n}}{\sqrt{n}-\sqrt{2}}.$$

If this inequality holds for infinitely many $n$ then

$$c \leq \lim_{n\to\infty} \frac{\sqrt{2}\sqrt{n}}{\sqrt{n}-\sqrt{2}} = \sqrt{2}.$$

This completes our solution.

**Remark.** In our solution above, one could try to take $M \approx u\sqrt{n}$ and $N \approx w\sqrt{n}$ for some $u, w \geq 0$. It is a nice excercise to see that the best estimate for $c$ is achieved wehen $u = w = \sqrt{2}$.

The solution submitted by Josiah Moltz is essentially based on the same idea, but the details of how he handles the estimates is different and very interesting (though more complicated), so we now outline his approach.

**Outline of the solution by Josiah Moltz.** Josiah starts by observing that

$$-\ln\left(1-\frac{1}{x}\right) \geq \frac{1}{x}$$

for $x > 1$ (we leave this as a simple calculus excercise). It follows that

$$\ln\left(\frac{k}{k-1}\right) \geq \frac{1}{k}$$

for all integers $k \geq 2$. Thus, the condition (ii) implies that

$$|a_k - a_{k-1}| \leq \ln\left(\frac{k}{k-1}\right) \tag{5}$$

for $k \geq 2$.
Suppose now that $|a_n| \geq c/\sqrt{n}$ for infinitely many $n$. Replacing the sequence $a_1, a_2, \ldots$ with the sequence $-a_1, -a_2, \ldots$ if necessary, we may assume that $a_n \geq c/\sqrt{n}$ for infinitely many $n$. Let $T$ be the set of all such $n$ and let $n \in T$. We see that

$$a_{n+i}-a_n = (a_{n+1}-a_n)+(a_{n+2}-a_{n+1})+\ldots+(a_{n+i}-a_{n+i-1}) \geq -\ln\left(\frac{n+1}{n}\right)-\ldots-\ln\left(\frac{n+i}{n+i-1}\right) = -\ln\left(\frac{n+i}{n}\right)$$

(this is why working with $\ln(k/(k-1))$ is more convenient than working with $1/k$). In other words,

$$a_{n+i} \geq a_n - \ln\left(\frac{n+i}{n}\right) \geq \frac{c}{\sqrt{n}} - \ln\left(\frac{n+i}{n}\right) \tag{6}$$

for every integer $i \geq 0$. In the same way we show that

$$a_{n-i} \geq \frac{c}{\sqrt{n}} - \ln\left(\frac{n}{n-i}\right) \tag{7}$$

for every integer $0 \leq i < n$. Let

$$R_n(x) = \frac{c}{\sqrt{n}} - \ln\left(\frac{x}{n}\right) \text{ and } L_n(x) = \frac{c}{\sqrt{n}} - \ln\left(\frac{n}{x}\right).$$

The function $R_n(x)$ is decreasing and the function $L_n(x)$ is increasing. Also $R_n(u) = 0$ for $u = ne^{c/\sqrt{n}}$, so $R_n(x) \geq 0$ for $x \leq ne^{c/\sqrt{n}}$. Similarly, $L_n(w) = 0$ for $w = ne^{-c/\sqrt{n}}$ and $L_n(x) \geq 0$ for $x \geq ne^{-c/\sqrt{n}}$. The inequality (6) yileds

$$a_{n+i} \geq R_n(n+i) = \int_{n+i}^{n+i+1} R_n(n+i)\mathrm{d}t \geq \int_{n+i}^{n+i+1} R_n(t)\mathrm{d}t$$

(for the last inequality we use that $R_n$ is decreasing). Similarly, using (7) we get

$$a_{n-i} \geq L_n(n-i) \geq \int_{n-i-1}^{n-i} L_n(t)\mathrm{d}t$$

Adding these inequalities we get

$$\sum_{i=0}^{M} a_{n+i} \geq \sum_{i=0}^{M} \int_{n+i}^{n+i+1} R_n(t)\mathrm{d}t = \int_{n}^{n+M+1} R_n(t)\mathrm{d}t$$

and

$$\sum_{i=1}^{N} a_{n-i} \geq \sum_{i=1}^{N} \int_{n-i-1}^{n-i} L_n(t)\mathrm{d}t = \int_{n-N-1}^{n-1} L_n(t)\mathrm{d}t = \int_{n-N-1}^{n} L_n(t)\mathrm{d}t - \int_{n-1}^{n} L_n(t)\mathrm{d}t$$

(for now, $M$ and $N$ are unspecified positive integers). As in our first solution, we note now that condition (i) implies that

$$\sum_{i=n-N}^{n+M} a_i = \sum_{i=1}^{n+M} a_i - \sum_{i=1}^{n-N-1} a_i \leq 2.$$

Thus

$$2 \geq \sum_{i=n-N}^{n+M} a_i = \sum_{i=0}^{M} a_{n+i} + \sum_{i=1}^{N} a_{n-i} \geq \int_{n}^{n+M+1} R_n(t)\mathrm{d}t + \int_{n-N-1}^{n} L_n(t)\mathrm{d}t - \int_{n-1}^{n} L_n(t)\mathrm{d}t.$$

Since $R_n(t)$ is a decreasing function and $R_n(u) = 0$ (recall that $u = ne^{c/\sqrt{n}}$) the integral

$$I(z) = \int_{n}^{n+z} R_n(t)\mathrm{d}t$$

attains its maximum for $z = u$. This suggests to take $M = \lfloor u - n \rfloor$ so that $u < n + M + 1 \leq u + 1$. Similarly, since $L_n(t)$ is an increasing function and $L_n(w) = 0$ (recall that $w = ne^{-c/\sqrt{n}}$) the integral

$$J(z) = \int_{n-z}^{n} L_n(t)\mathrm{d}t$$

attains its maximum for $z = w$. This suggest to take $N = \lfloor n - w \rfloor$ so that $w - 1 \leq n - N - 1 < w$. With these choices of $M$ and $N$ we get

$$2 \geq \int_{n}^{u} R_n(t)\mathrm{d}t + \int_{w}^{n} L_n(t)\mathrm{d}t + \int_{u}^{n+M+1} R_n(t)\mathrm{d}t + \int_{n-N-1}^{w} L_n(t)\mathrm{d}t - \int_{n-1}^{n} L_n(t)\mathrm{d}t. \qquad (8)$$

Recall that this inequality holds for every $n$ in the infinite set $T$ of indeces such that $a_n \geq c/\sqrt{n}$. We are going to show now that when $n$ tends to infinity each of the first two intgerals in (8) converges to $c^2/2$ and the last three integrals all converge to 0. This implies that $2 \geq c^2/2 + c^2/2 = c^2$, i.e. $c \leq \sqrt{2}$. To this end note that $R_n(x)$ is the derivative of $(\frac{c}{\sqrt{n}} + \ln n + 1)x - x\ln x$ and $L_n(x)$ is the derivative of $(\frac{c}{\sqrt{n}} - \ln n - 1)x + x\ln x$. It is now a simple calculation to get

$$\int_{n}^{u} R_n(t)\mathrm{d}t = n\left(e^{c/\sqrt{n}} - 1 - \frac{c}{\sqrt{n}}\right),$$

$$\int_{w}^{n} L_n(t)\mathrm{d}t = n\left(e^{-c/\sqrt{n}} - 1 + \frac{c}{\sqrt{n}}\right),$$

$$\int_{n-1}^{n} L_n(t)\mathrm{d}t = \frac{c}{\sqrt{n}} - 1 + (n-1)\ln\frac{n}{n-1} = \frac{c}{\sqrt{n}} + (n-1)\ln\left(1 + \frac{1}{n-1}\right) - 1.$$

A simple calculus excercise shows that

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}$$

from which we easily deduce that

$$\lim_{n\to\infty} n\left(e^{c/\sqrt{n}} - 1 - \frac{c}{\sqrt{n}}\right) = \frac{c^2}{2} = \lim_{n\to\infty} n\left(e^{-c/\sqrt{n}} - 1 + \frac{c}{\sqrt{n}}\right)$$

(take $x = \pm c/\sqrt{n}$ so $n = c^2/x^2$). Another simple calculus excercise shows that

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

so

$$\lim_{n\to\infty} (n-1)\ln\left(1 + \frac{1}{n-1}\right) = 1$$

(take $x = 1/(n-1)$) and therefore

$$\lim_{n\to\infty} \left(\frac{c}{\sqrt{n}} + (n-1)\ln\left(1 + \frac{1}{n-1}\right) - 1\right) = 0.$$

Since $R_n$ is decreasing, $u < n + M + 1 \le u + 1$, and $R_n(u) = 0$, we have

$$\left|\int_u^{n+M+1} R_n(t)\mathrm{d}t\right| \le |R_n(u+1)| = |R_n(u+1) - R_n(u)| = |R_n'(s)|$$

for some $u < s < u + 1$ (by the Mean Value Theorem). Since $R_n'(x) = -1/x$, we get that

$$\left|\int_u^{n+M+1} R_n(t)\mathrm{d}t\right| \le \frac{1}{u} \le \frac{1}{n}$$

and therefore

$$\lim_{n\to\infty} \int_u^{n+M+1} R_n(t)\mathrm{d}t = 0.$$

In a similar way we show that

$$\left|\int_{n-N-1}^{w} L_n(t)\mathrm{d}t\right| \le \frac{1}{w-1} = \frac{1}{ne^{-c/\sqrt{n}} - 1}.$$

Using the simple fact that $e^x \ge 1 + x$ for all $x$ we easily show that

$$\lim_{n\to\infty} \frac{1}{ne^{-c/\sqrt{n}} - 1} = 0$$

and therefore

$$\lim_{n\to\infty} \int_{n-N-1}^{w} L_n(t)\mathrm{d}t = 0.$$

This completes our solution.

**Remark.** Josiah's solution shows a stronger result, where condition (ii) is replaced by (5).

**Remark.** Looking more carefully at how we got the stimates for $a_{n\pm i}$ in terms of integrals of $R_n$ and $L_n$, we easily see that if $R_n(n+i) \ge 0$ and $L_n(n-j) \ge 0$ then

$$a_{n+i} \ge R_n(n+i) \ge \int_{n+i}^{n+i+z} R_n(t)\mathrm{d}t \text{ end } a_{n-j} \ge L_n(n-j) \ge \int_{n-j-z}^{n-j} L_n(t)\mathrm{d}t$$

for any $z \in [0,1]$. This easily allows to improve (8) to the inequality:

$$2 \ge \int_n^u R_n(t)\mathrm{d}t + \int_w^n L_n(t)\mathrm{d}t - \int_{n-1}^n L_n(t)\mathrm{d}t.$$

This slightly simplifies the argument making the computations of the last 2 limits of the solution unnecessary.

**Remark.** It is not hard to see that for large $n$ both $M$ and $N$ in Josiah's solution are approximately $c\sqrt{n}$.

**Problem.** The result of the problem can be formulated as follows:

$$\limsup_{n\to\infty} \sqrt{n}|a_n| \leq \sqrt{2}.$$

We do not know if there is a better upper bound for $\limsup_{n\to\infty} \sqrt{n}|a_n|$.

**Excercise.** Suppose that a sequence $f_1, f_2, \ldots$ of positive numbers converges to 0. A sequence $a_1, a_2, \ldots$ of real numbers has the following properties:

(i) $|a_1 + a_2 + \ldots + a_k| \leq 1$ for every $k$;

(ii) $|a_k - a_{k-1}| \leq f_k$ for every $k \geq 2$.

Prove that $\lim_{n\to\infty} a_n = 0$.