Problem 2. Is it possible to find two functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ and a rational function $S(u, w)$ in two variables such that

$$\cos(xy) = S(f(x), g(y))$$

for all real numbers x, y. (Recall that a rational function in 2 variables is a quotient of two polynomial functions in two variables).

Solution. We will show that there is no such representation of the function $\cos(xy)$. Suppose by the way of contradiction that such a representation exists. Let $S(u, w) = F(u, w)/G(u, w)$ for some polynomials F, G.

Note that if $f(a) = f(b)$ for some a, b then $\cos(ay) = \cos(by)$ for all y. Recall that $\cos(-u) = \cos(u)$ for every u. Thus $\cos(|a|y) = \cos(|b|y)$ for all y. If $|a| < |b|$ we take $y = 1/|b|$ to see that $\cos(|a|/|b|) = \cos 1$ and $0 \leq |a|/|b| < 1 < \pi/2$, which contradicts the fact that cos is a one-to-one function on $[0, \pi/2]$ Thus $|a| < |b|$ is not possible. Similarly $|a| > |b|$ is not possible. Thus $|a| = |b|$, i.e. $a = \pm b$. It follows that f must be one-to-one on $[0, \infty)$. In particular, the numbers $f(\pi/2 + k\pi)$, $k = 0, 1, \ldots$ are all distinct. Now

$$0 = \cos(\pi/2 + k\pi) = \cos((\pi/2 + k\pi) \cdot 1) = S(f(\pi/2 + k\pi), g(1)) = \frac{F(f(\pi/2 + k\pi), g(1))}{G(f(\pi/2 + k\pi), g(1))}.$$

Thus $F(f(\pi/2 + k\pi), g(1)) = 0$ for every k. Note that $F(u, g(1))$ is a polynomial in one variable u and each number $f(\pi/2 + k\pi)$ is a root of this polynomial. Since a non-zero polynomial in one variable can have only finitely many distinct roots, we conculde that $F(u, g(1)) = 0$ for all u. But then

$$\cos(x) = S(f(x), g(1)) = \frac{F(f(x), g(1))}{G(f(x), g(1))} = 0$$

for every x, which is clearly incorrect. Therefore the representation in question does not exist.