Problem 1. Consider a convex polyhedron whose every face is a triangle. We color the vertices of the polyhedron in 3 colors. Prove that the number of faces of the polyhedron whose vertices are of all three colors is even.

Solution. We call en edge of the polyhedron *colorful* if its vertices have different colors. Let t be the number of colorful edges of the polyhedron. Note that any face of the polyhedron has either 0, 2, or 3 colorful edges. Let t_i be the number of colorful edges of the face i. Let F be the collection of all faces, and T the subset of F of all faces which have vertices of all three colors. Thus the problem asks us to prove that the number $|T|$ of elements of T is even. Note that a face i of the polyhedron is in T if and only if $t_i = 3$.

If we add the numbers f_i for all faces, we will count each colorful edge twice (as every edge belongs to exactly two faces). This means that

$$\sum_{i \in F} t_i = 2t.$$

If $i \notin T$ then t_i is either 0 or 2, i.e. t_i is even. It follows that the numbers $\sum\limits_{i \in F} t_i$ and $\sum\limits_{i \in T} t_i$ differ by an even number, i.e. they have the same parity. Since $\sum\limits_{i \in F} t_i = 2t$ is even, we conclude that $\sum\limits_{i \in T} t_i = 3|T|$ is even. Consequently the number $|T|$ is also even, i.e. T has an even number of elements.

Exercise. Prove that the number of faces with an odd number of sides in a convex polyhedron is always even.