Problem 5. Let

$$a_n = \sqrt[n^2]{\prod_{i=1}^{n}\prod_{j=1}^{n}\gcd(i,j)},$$

where $\gcd(i,j)$ is the greatest common divisor of i and j. Prove that the sequence a_n converges.

Solution. Let $P_n = \prod_{i=1}^{n}\prod_{j=1}^{n}\gcd(i,j)$. We will investigate the prime power factorization of P_n. Any prime divisor of P_n is smaller or equal than n. Thus

$$P_n = \prod_{p\le n} p^{E_p(n)},$$

for some non-negative integers $E_p(n)$ (here p runs through prime numbers $\le n$). If $e_{ij}(p)$ is the largest integer such that $p^{e_{ij}(p)}$ divides $\gcd(i,j)$ then

$$E_p(n) = \sum_{i=1}^{n}\sum_{j=1}^{n} e_{ij}(p).$$

Let us now count the number of pairs (i,j) such that $1 \le i,j \le n$ and $p^k|\gcd(i,j)$. This means that $i = p^k a$, $j = p^k b$ for some positive integers a,b both $\le n/p^k$. Thus we have $\left\lfloor \frac{n}{p^k} \right\rfloor$ choices for each a and b and therefore there are $\left\lfloor \frac{n}{p^k} \right\rfloor^2$ pairs (i,j) such that $1 \le i,j \le n$ and $p^k|\gcd(i,j)$ (recall that $\lfloor a \rfloor$ denotes the largest integer which does not exceed a and it is called the floor of a). In other words, the number of pairs (i,j) such that $1 \le i,j \le n$ and $e_{ij}(p) \ge k$ is $\left\lfloor \frac{n}{p^k} \right\rfloor^2$. It follows that number of pairs (i,j) such that $1 \le i,j \le n$ and $e_{ij}(p) = k$ is

$$\left\lfloor \frac{n}{p^k} \right\rfloor^2 - \left\lfloor \frac{n}{p^{k+1}} \right\rfloor^2.$$

Consequently,

$$E_p(n) = \sum_{i=1}^{n}\sum_{j=1}^{n} e_{ij}(p) = \sum_{k=1}^{\infty} k\left(\left\lfloor \frac{n}{p^k} \right\rfloor^2 - \left\lfloor \frac{n}{p^{k+1}} \right\rfloor^2\right) = \sum_{k=1}^{\infty}\left\lfloor \frac{n}{p^k} \right\rfloor^2. \tag{1}$$

The last equality follows from the following useful identity

$$(a_1 - a_2) + 2(a_2 - a_3) + 3(a_3 - a_4) + \ldots = a_1 + a_2 + a_3 + \ldots$$

provided $a_k = 0$ for all sufficiently large k (note that $\left\lfloor \frac{n}{p^k} \right\rfloor = 0$ for all sufficiently large k).

Remark. The evaluation of $E_p(n)$ is similar to the well known formula for the prime factorization of $n!$, which says that the highest power of p which divides $n!$ is p^e, where

$$e = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots$$

Returning to our solution, note that

$$\ln a_n = \frac{1}{n^2}\ln P_n = \frac{1}{n^2}\sum_{p\le n} E_p(n)\ln p.$$

It suffices to show that the sequence $\ln a_n$ converges. We need to estimate the numbers $E_p(n)$. Note that for any $a > 0$ we have $a^2 - 2a < \lfloor a \rfloor^2 \le a^2$. Thus

$$E_p(n) = \sum_{k=1}^{\infty}\left\lfloor \frac{n}{p^k} \right\rfloor^2 \le \sum_{k=1}^{\infty}\left(\frac{n}{p^k}\right)^2 = \frac{n^2}{p^2-1}$$

and

$$E_p(n) \geq \sum_{k=1}^{\infty}\left(\frac{n}{p^k}\right)^2 - 2\sum_{k=1}^{\infty}\frac{n}{p^k} = \frac{n^2}{p^2-1} - \frac{2n}{p-1}.$$

We see that

$$\ln a_n \leq \frac{1}{n^2}\sum_{p\leq n}\frac{n^2}{p^2-1}\ln p = \sum_{p\leq n}\frac{\ln p}{p^2-1}$$

and

$$\ln a_n \geq \frac{1}{n^2}\sum_{p\leq n}\left(\frac{n^2}{p^2-1} - \frac{2n}{p-1}\right)\ln p = \sum_{p\leq n}\frac{\ln p}{p^2-1} - \frac{2}{n}\sum_{p\leq n}\frac{\ln p}{p-1}.$$

Set $B_n = \sum_{p\leq n}\frac{\ln p}{p^2-1}$ and $C_n = \frac{2}{n}\sum_{p\leq n}\frac{\ln p}{p-1}$. Then we have

$$B_n - C_n \leq \ln a_n \leq B_n.$$

If we can show that B_n converges and C_n converges to 0, then $\ln a_n$ converges to the same limit as B_n by the pinching theorem. We have

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty}\sum_{p\leq n}\frac{\ln p}{p^2-1} = \sum_{p \text{ a prime}}\frac{\ln p}{p^2-1}$$

and the last series converges by the comparison test with the convergent series $\sum_n \frac{\ln n}{n^2-1}$. Furthermore,

$$0 \leq C_n = \frac{2}{n}\sum_{p\leq n}\frac{\ln p}{p-1} \leq \frac{2\ln n}{n}\sum_{k=1}^{n}\frac{1}{k} \leq \frac{2\ln n(1+\ln n)}{n}$$

(we use here the inequality $\sum_{k=2}^{n}\frac{1}{k} \leq \int_1^n \frac{dt}{t} = \ln n$). Since the sequence $u_n = \frac{2\ln n(1+\ln n)}{n}$ converges to 0, we see that $\lim_{n\to\infty} C_n = 0$ (by the pinching theorem). This completes our proof that $\ln a_n$ converges, and as a consequence we see that

$$\lim_{n\to\infty} a_n = e^{\sum_p \frac{\ln p}{p^2-1}}$$

(where the sum in the exponent is over all prime numbers).

Second solution. We have

$$\ln a_n = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\ln\gcd(i,j).$$

We want to count the number of pairs (i,j) such that $1 \leq i,j \leq n$ and $\gcd(i,j) = d$. Let $h(x)$ be the number of pairs (a,b) of integers such that $1 \leq a,b \leq x$ and $\gcd(a,b) = 1$. Since $\gcd(i,j) = d$ means that $i = da$, $j = db$ and $1 \leq a,b \leq n/d$ and $\gcd(a,b) = 1$, we see that the number of pairs (i,j) such that $1 \leq i,j \leq n$ and $\gcd(i,j) = d$ is equal to $h(n/d)$. Thus

$$\ln a_n = \frac{1}{n^2}\sum_{d=1}^{n}(\ln d)h(n/d) = \sum_{d=1}^{n}\frac{\ln d}{d^2}\frac{h(n/d)}{(n/d)^2} = \sum_{d\leq n}\frac{\ln d}{d^2}\frac{h(n/d)}{(n/d)^2}. \tag{2}$$

We need to get some understanding of the function $h(x)/x^2$. To this end, we recall that there is a unique function $\mu : \mathbb{N} \longrightarrow \mathbb{Z}$ such that

$$\sum_{d|n}\mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}.$$

The function μ is called the Möbius function and it is not hard to see that if $n = p_1p_2\dots p_k$ is a product of k distinct primes then $\mu(n) = (-1)^k$ and that $\mu(n) = 0$ for all other $n > 1$. We can use μ to express h as follows:

$$h(x) = \sum_{a\leq x}\sum_{b\leq x}\sum_{d|\gcd(a,b)}\mu(d).$$

Now the number of pairs (a,b) such that a given d divides $\gcd(a,b)$ (i.e. divides both a and b) is equal to $\left\lfloor \frac{x}{d} \right\rfloor^2$ (we just count all pairs of integers between 1 and x/d). Thus the formula for $h(x)$ can be rewritten as

$$h(x) = \sum_{d\leq x} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor^2.$$

It follows that

$$\frac{h(x)}{x^2} = \sum_{d\leq x} \frac{\mu(d)}{d^2} \frac{\left\lfloor \frac{x}{d} \right\rfloor^2}{(\frac{x}{d})^2}. \tag{3}$$

We need now the following simple but useful result.

Proposition. *Let* $\sum_{n=1}^{\infty} u_n$ *be an absolutely convergent series and let* $\psi : (0,\infty) \longrightarrow \mathbb{R}$ *be a bounded function such that* $\lim_{x\to\infty} \psi(x) = L$ *exists. Then*

$$\lim_{x\to\infty} \sum_{d\leq x} u_d \psi(x/d) = L \sum_{n=1}^{\infty} u_n.$$

We will first use the proposition to complete our solution and then discuss the proposition itself. Let $u_n = \mu(n)/n^2$ and $\psi(x) = \frac{\lfloor x \rfloor^2}{x^2}$. Clearly $\lim_{x\to\infty} \psi(x) = 1$ and $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}$ converges absolutely (since $|\mu(n)| \in \{0,1\}$ for all n). It follows from the proposition that

$$\lim_{x\to\infty} \frac{h(x)}{x^2} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2}. \tag{4}$$

We will now apply the proposition to formula (2), i.e. we take $u_n = \ln n/n^2$ and $\psi(x) = h(x)/x^2$. Clearly $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges absolutely. By (4) and the proposition, we see that

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \sum_{d\leq n} \frac{\ln d}{d^2} \frac{h(n/d)}{(n/d)^2} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{n=1}^{\infty} \frac{\ln n}{n^2}.$$

This completes our proof that a_n converges.

It remains to justify the proposition. Let $G = \sum_{n=1}^{\infty} |u_n|$ and let $|\psi(x)| \leq B$ for all x. Let $\eta(x) = \sup_{t\geq x} |\psi(x) - L|$. Then $\lim_{x\to\infty} \eta(x) = 0$. Note that

$$0 \leq \left| \sum_{d\leq x} u_d \psi(x/d) - L \sum_{n=1}^{\infty} u_n \right| = \left| \sum_{d\leq\sqrt{x}} u_d(\psi(x/d) - L) + \sum_{\sqrt{x}<d\leq x} u_d(\psi(x/d) - L) + L \sum_{n>x} u_n \right| \leq$$

$$\sum_{d\leq\sqrt{x}} |u_d||\psi(x/d) - L| + \sum_{\sqrt{x}<d\leq x} |u_d||\psi(x/d) - L| + |L| \sum_{n>x} |u_n| \leq$$

$$\eta(\sqrt{x})G + (B + |L|) \sum_{\sqrt{x}<d\leq x} |u_d| + |L| \sum_{n>x} |u_n| \leq \eta(\sqrt{x})G + (B + |L|) \sum_{n>\sqrt{x}} |u_n|.$$

Since

$$\lim_{x\to\infty} \left(\eta(\sqrt{x})G + (B + |L|) \sum_{n>\sqrt{x}} |u_n| \right) = 0,$$

we see by the pinching theorem that

$$\lim_{x\to\infty} \left(\sum_{d\leq x} u_d \psi(x/d) - L \sum_{n=1}^{\infty} u_n \right) = 0,$$

which is exactly the conclusion of the proposition.

We end the discussion with several exercises related to the methods employed in the above solutions.

Exercise 1. Let $\phi(m)$ be the number of positive integers smaller or equal than m and relatively prime to m (ϕ is called the Euler function). Show that

$$h(x) = -1 + 2\sum_{d\leq x} \phi(d).$$

Show also that

$$\sum_{d|n} \phi(d) = n.$$

Hint: Count the number of simple fractions in $(0, 1]$ whose denominator divides n.

Exercise 2. For a given sequence $f_1, f_2, \ldots$ define a new sequence $g_1, g_2, \ldots$ by the formula

$$g_n = \sum_{d|n} f_d.$$

a) Prove that

$$f_n = \sum_{d|n} g_d \mu(n/d),$$

where μ is the Möbius function. This is called the Möbius inversion formula. Conclude that

$$\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}.$$

b) Show that

$$\sum_{d\leq x} g_d = \sum_{d\leq x}\sum_{k\leq x/d} f_k \quad \text{and} \quad \sum_{d\leq x} f_d = \sum_{d\leq x} \mu(d) \sum_{k\leq x/d} g_k.$$

c) Show that if $k > 1$ and $\lim_{x\to\infty} \frac{1}{x^k}\sum_{d\leq x} f_d = L$ exists then

$$\lim_{x\to\infty} \frac{1}{x^k}\sum_{d\leq x} g_d = L\sum_{n=1}^{\infty} \frac{1}{n^k}.$$

d) Show that if $k > 1$ and $\lim_{x\to\infty} \frac{1}{x^k}\sum_{d\leq x} g_d = K$ exists then

$$\lim_{x\to\infty} \frac{1}{x^k}\sum_{d\leq x} f_d = K\sum_{n=1}^{\infty} \frac{\mu(n)}{n^k}.$$

e) Use c) and d) to show that for any $k > 1$ we have

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^k}\sum_{n=1}^{\infty} \frac{1}{n^k} = 1.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$ we conclude that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{6}{\pi^2}.$$

Exercise 3. Comparing the the formulas for $\lim_{n\to\infty} \ln a_n$ from our first and second solutions we see that

$$\sum_{p \text{ a prime}} \frac{\ln p}{p^2 - 1} = \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{\ln n}{n^2}.$$

Give a direct proof of this equality.