Problem 2. Find all positive integers n which have the following property: there is a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for every real number t the equation $f(x) = t$ has either no solutions or exactly n different solutions.

Solution. We show first that for every odd natural number n there is a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that the equation $f(x) = t$ has exactly n different solutions for every real number t. Let n be odd. Define $h_n : [0,1] \longrightarrow [0,1]$ as follows:

$$h_n(x) = \begin{cases} nx - k, & \text{for } 0 \le k < n \text{ even and } \frac{k}{n} \le x \le \frac{k+1}{n} \\ k - nx, & \text{for } 0 \le k < n \text{ odd and } \frac{k}{n} \le x \le \frac{k+1}{n}. \end{cases}$$

It is easy to see that $h_n(0) = 0$, $h_n(1) = 1$, $h_n(x) = t$ has exactly n solutions for $0 < t < 1$ (one solution in each interval $(k/n, (k+1)/n$, $k = 0, 1, \ldots, n-1)$, and each of the equations $h_n(x) = 0$ and $h_n(x) = 1$ has exactly $(n+1)/2$ solutions. Now define f by the formula

$$f(x) = \lfloor x \rfloor + h_n(x - \lfloor x \rfloor),$$

where $\lfloor x \rfloor$ is the floor of x, i.e. the largest integer smaller or equal than x. Note that $f(x+m) = f(x)+m$ for any integer m. It is now easy to see that f has the required property (the reader should draw the graph of f for $n = 3, 5$).

We claim that for $n = 2k$ even there are no functions with the required property. Assume contrary, that such a function f exists. We will show that this leads to a contradiction. Let s be a value of the function f. Thus there are real numbers $u_1 < u_2 < \ldots < u_{2k}$ such that $f(u_1) = \ldots = f(u_{2k}) = s$. There are no other x such that $f(x) = s$. In particular, s is not a value of f on any of the $2k+1$ intervals $(-\infty, u_1)$, (u_{2k}, ∞), (u_i, u_{i+1}), $i = 1, \ldots, 2k-1$. It follows that on each of these intervals f is either smaller than s or larger than s (since f is continuous; we use the intermediate value theorem). Thus among the $2k-1$ intervals (u_i, u_{i+1}) there are k intervals on which f is smaller than s, or k intervals on which f is larger than s. Without loss of generality we may assume f is larger than s on the intervals $I_j = (u_{i_j}, u_{i_j+1})$ for some $1 \le i_1 < i_2 < \ldots < i_k < 2k$. Let M_j be the largest value of f on I_j (it exits since f is continuous on the closed interval I_j). Clearly $M_j > s$ for $j = 1, \ldots, k$. Take $y_j \in I_j$ such that $f(y_j) = M_j$. If $s < t < \min(M_1, ..., M_k)$ then f assumes value t at least twice on each interval I_j (at least once on each (u_{i_j}, y_j) and (y_j, u_{i_j+1}). This gives at lest $2k$ points at which f assumes value t. It follows that t is not a value of f on any of the remaining $k+1$ intervals. This implies that f is less than s on the remaining intervals. Now let $M = \max(M_1, ..., M_k)$. There are $2k$ points $z_1 < \ldots < z_{2k}$ such that $f(z_i) = M$ and each z_i is in the interior of one of the intervals I_j. This implies that f has a local maximum (actually a global maximum) equal to M at each z_i. Now pick $\epsilon > 0$ such that $z_i + \epsilon < z_{i+1} - \epsilon$ for all i. Each value $f(z_i \pm \epsilon)$ is smaller than M. Let t be any number smaller than M and larger than each $f(z_i \pm \epsilon)$. Then f must assume value t on each interval $(z_i - \epsilon, z_i)$ and $(z_i, z_i + \epsilon)$, $i = 1, \ldots, 2k$. These $4k$ intervals are pairwise disjoint, so $f(x) = t$ has at least $4k$ solutions, contrary to our assumption.

The following more challenging problem seams a natural follow-up question.

Problem. Find all positive integers n which have the following property: there is a continuous function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that for every pair of real number $t = (t_1, t_2)$ the equation $f(x) = t$ has either no solutions or exactly n different solutions.