Problem 6. Let A and B be square matrices of the same size such that $A^{2020} = I = B^{2020}$ and $AB = -BA$. Prove that $I + A + B$ is invertible. (Here I is the identity matrix).

Solution. Note that the condition $AB = -BA$ implies that $A^2B = BA^2$. Suppose that $I + A + B$ is not invertible. Then there is a non-zero vector $\mathbf{v}$ such that $(I + A + B)\mathbf{v} = 0$. In other words, $(I + A)\mathbf{v} = -B\mathbf{v}$. Multiplying by B we get

$$-B^2\mathbf{v} = B(I + A)\mathbf{v} = (I - A)B\mathbf{v} = (I - A)(-(I + A)\mathbf{v}) = -(I - A^2)\mathbf{v}$$

so $B^2\mathbf{v} = (I - A^2)\mathbf{v}$. Multiplying by B again, we get

$$B^3\mathbf{v} = B(I - A^2)\mathbf{v} = (I - A^2)B\mathbf{v} = -(I - A^2)(I + A)\mathbf{v}.$$

Continuing this process we see that for any integer $k \geq 0$ we have

$$B^{2k}\mathbf{v} = (I - A^2)^k\mathbf{v} \quad \text{and} \quad B^{2k+1}\mathbf{v} = -(I - A^2)^k(I + A)\mathbf{v}.$$

Taking $k = 1010$, we have $\mathbf{v} = B^{2020}\mathbf{v} = (I - A^2)^{1010}\mathbf{v}$. It follows that 1 is an eigenvalue of $(I - A^2)^{1010}$ with eigenvector $\mathbf{v}$. Thus there is an eigenvalue w of $I - A^2$ such that $w^{1010} = 1$. Any eigenvalue of $I - A^2$ is of the form $1 - u^2$ for some eigenvalue u of A. Thus $w = 1 - u^2$ for some eigenvalue u of A. Since $A^{2020} = I$, we have $u^{2020} = 1$. In particular, both w and $1 - w = u^2$ have absolute value 1. So $w\overline{w} = 1$ and

$$1 = (1 - w)(\overline{1 - w}) = (1 - w)(1 - \overline{w}) = 1 - w - \overline{w} + w\overline{w} = 2 - w - \overline{w}.$$

We see that $\overline{w} = w^{-1}$ and $w + w^{-1} = 1$. Thus $w^2 - w + 1 = 0$, so $w^3 = -1$ (recall that $x^3 + 1 = (x + 1)(1 - x + x^2)$). Since $3|1011$, we have $w^{1011} = -1$. Since $w^{1010} = 1$, we conclude that $w = -1$, which is not possible (as $w + w^{-1} = 1$). The contradiction shows that $I + A + B$ must be invertible.

Challenge. Find matrices A, B which satisfy the conditions of the problem.