Problem 4. a) Let $x_1, \dots, x_n$ be real numbers. Prove that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\sin(x_i - x_j)}{x_i - x_j} \geq \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\sin(x_i + x_j)}{x_i + x_j}$$

with the convention that $\dfrac{\sin x}{x} = 1$ when $x = 0$.

b) Compute $\displaystyle\int_0^1 \sin 2x \sin 5x \; \mathrm{d}x$.

Solution. We start with part b) as it will give us a clue how to solve a).
Recall that $\cos u - \cos w = -2\sin\frac{(u+w)}{2}\sin\frac{(u-w)}{2}$. Setting $u = a-b$, $w = a+b$, we get

$$2\sin a \sin b = \cos(a-b) - \cos(a+b).$$

Thus

$$2\int_0^1 \sin ax \sin bx \; \mathrm{d}x = \int_0^1 \cos(x(a-b)) \; \mathrm{d}x - \int_0^1 \cos(x(a+b)) \; \mathrm{d}x = \frac{\sin(a-b)}{a-b} - \frac{\sin(a+b)}{a+b}. \quad (1)$$

Note that (1) works for $a = b$ too.

In particular, $\displaystyle\int_0^1 \sin 2x \sin 5x \; \mathrm{d}x = \frac{\sin 3}{6} - \frac{\sin 7}{14}$.

We can now solve a). Taking (1) with $a = x_i$, $b = x_j$ yields

$$\frac{\sin(x_i - x_j)}{x_i - x_j} - \frac{\sin(x_i + x_j)}{x_i + x_j} = 2\int_0^1 \sin tx_i \sin tx_j \; \mathrm{d}t.$$

Adding these equalities, we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\sin(x_i - x_j)}{x_i - x_j} - \frac{\sin(x_i + x_j)}{x_i + x_j}\right) = 2\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^1 \sin tx_i \sin tx_j \; \mathrm{d}t =$$

$$2\int_0^1 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \sin tx_i \sin tx_j\right) \mathrm{d}t = 2\int_0^1 \left(\sum_{i=1}^{n} \sin tx_i\right)^2 \mathrm{d}t \geq 0$$

which proves the inequality in part a).

A (slightly) different solution. We now show a solution which does not use integration. Consider the function

$$g(t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\sin t(x_i - x_j)}{x_i - x_j} - \frac{\sin t(x_i + x_j)}{x_i + x_j}\right)$$

with the convention that $\sin ta/a = t$ when $a = 0$. We need to prove that $g(1) \geq 0$. Note that $g(0) = 0$ and

$$g'(t) = \sum_{i=1}^{n}\sum_{j=1}^{n}(\cos t(x_i - x_j) - \cos t(x_i + x_j)) = \sum_{i=1}^{n}\sum_{j=1}^{n} 2\sin tx_i \sin tx_j = 2\left(\sum_{i=1}^{n}\sin tx_i\right)^2 \geq 0.$$

Thus the function g is nondecreasing and therefore $g(1) \geq g(0) = 0$.

Exercise. When does the equality hold in part a)?