Math 226 **(White)** **Midterm Exam**
Integration Techniques and Applications **April 20, 2016**

Do NOT turn over this page until instructed to begin.

Name: ____

Section:____

Binghamton ID Number:____

Instructions

Write clear, careful, neat solutions to the questions in the space provided.

No books, no notes, no electronic devices (calculators, cell phones, smart watches, etc.) allowed!

Write all your work on the test–nothing else will be graded. **You must show all your work and justify your answers.** Solutions presented without sufficient supporting work may receive no credit. Your work must be legible, and numerical answers should be presented as exact mathematical expressions, simplified as appropriate, and not by a decimal approximation, unless explicitly required by the problem.

On some problems you may be asked to use a specific method to solve the problem (for instance, "Use the Fundamental Theorem of Calculus to find..."). On all other problems, you may use any method we have covered. You may not use methods that we have not covered.

Wandering Eyes Policy

You must keep your eyes on your own work at all times. If you are found looking around, you will be warned once, and only once. A second infraction may result in automatic zero on this test, and possibly a referral to the Harpur College Academic Honesty Committee.

Duration of the Test

This is a timed test designed for one class period. You will start the test when your proctor tells you to start, and you MUST stop working when your proctor tells you to stop.

1	2	3	4	5	6	7	Total
24 pts	10 pts	24 pts	10 pts	15 pts	9 pts	8 pts	100 pts

1. (24 points) Find the most general antiderivatives and evaluate the following integrals. Be sure to show your work carefully, using correct notation.

(a) $\displaystyle\int \frac{\cos(\ln(x))}{x}\ dx$

(b) $\displaystyle\int_1^e \frac{x^5 - 2x + 1}{x}\ dx$

(c) $\displaystyle\int_0^{\pi/2} \frac{\sin(x)}{1+\cos^2(x)}\ dx$

(d) $\displaystyle\int \frac{1+x}{1+x^2}\ dx$

2. (10 points) Use the technique of "logarithmic differentiation" to compute the derivative of the function $f(x)$ below. Be sure to show your work.

$$f(x) = \frac{x^4 + 3x^2}{\cos^2(x)\sqrt{x}}.$$

3. (24 points) Compute the derivaties of the following functions. Be sure to show your work carefully, using correct notation.

(a) $\ln(\sin(x)\cos(x))$

(b) $5^{\arcsin(x)}$

(c) $\log_3(5x^4 + 7x^2 + 12)$

(d) $\arctan\left(\sqrt{x}\right)$

4. (10 points) A sum of $\$1,000$ is deposited in an account which earns interest with **continuous compounding**. After one year, the amount has increased to $\$1,200$.

(a) How much money will be in the account after 3 years? [This amount can be computed exactly without use of a calculator.]

(b) Write down an **exact mathematical expression** for the amount of time it takes for the account to be worth $\$10,000$.

5. (15 points) Compute the following limits. **Show the steps required to perform each computation.** If you use L'Hŏspital's Theorem, show where you use it and explain what type of limit you are using it on (i.e., which indeterminate form).

(a) $\lim_{x \to 0^+} (e^x + 3x)^{1/x}$

(b) $\lim_{x \to \infty} \frac{(\ln(x))^2}{x}$

(c) $\lim_{x \to 0} \frac{\sin(5x^3)}{\sin(3x^2)}$

6. (9 points)

(a) Suppose that $f(x)$ is a differentiable function with known inverse $g(x) = f^{-1}(x)$. Assume as well that $f(a) = b$, so that the point (a, b) lies on the graph of $f(x)$. Express the derivative $g'(b) = (f^{-1})'(b)$ in terms of the derivative of f. [**No work need be shown for this question.**]

(b) Explain (briefly) why the function $f(x) = \sin(x)$ does **not** have an inverse. What is required to define the function $\arcsin(x)$?

(c) Use the fact that $\sin(\pi/4) = 1/\sqrt{2}$ and your formula for (a) and explanation in (b) to find $\arcsin'(1/\sqrt{2})$.

7. (8 points) Compute the following integrals.

(a) $\int \arctan(x) \; dx$

(b) $\int x^2 e^x \; dx$

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Name: Solution Key

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1	2	3	4	5	6	7	Total
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1. (24 points) Find the most general antiderivatives and evaluate the following integrals. Be sure to show your work carefully, using correct notation.

(a) $\displaystyle\int \frac{\cos(\ln(x))}{x}\,dx = \int \cos(u(x)) \cdot u'(x)\,dx$

Substitution: $u(x) = \ln(x)$, $u'(x) = 1/x$

$$= \int \cos(u)\,du$$

$$= \sin(u(x)) + C$$

$$= \sin(\ln(x)) + C$$

(b) $\displaystyle\int_1^e \frac{x^5 - 2x + 1}{x}\,dx = \int_1^e x^4 - 2 + \frac{1}{x}\,dx$

$$= \frac{1}{5}x^5 - 2x + \ln(x)\Big]_1^e$$

$$= \left(\frac{1}{5}e^5 - 2e + \ln(e)\right) - \left(\frac{1}{5} - 2 + \ln(1)\right)$$

$$= \left(\frac{1}{5}e^5 - 2e + 1\right) - \left(\frac{1}{5} - 2\right)$$

$$= \frac{1}{5}e^5 - 2e + \frac{14}{5}$$

$$= \frac{1}{5}\left(e^5 - 10e + 14\right)$$

(c) $\int_0^{\pi/2} \frac{\sin(x)}{1+\cos^2(x)}\ dx$ $= -\int_0^{\pi/2} \frac{u'(x)}{1+u^2(x)}\, dx$

Substitution:

$u(x) = \cos(x)$

$u'(x) = -\sin(x)$

$\cos(0) = 1$

$\cos(\pi/2) = 0$

$= -\int_1^0 \frac{1}{1+u^2}\, du$

$= \int_0^1 \frac{1}{1+u^2}\, du$

$= \arctan(u) \Big]_0^1$

$= \arctan(1) - \arctan(0)$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}.$

(d) $\int \frac{1+x}{1+x^2}\ dx$ $= \int \frac{1}{1+x^2} + \frac{x}{1+x^2}\, dx$

Substitution:

$u(x) = x^2$

$u'(x) = 2x$

$= \int \frac{1}{1+x^2}\, dx + \int \frac{x}{1+x^2}\, dx$

$= \arctan(x) + \frac{1}{2}\int \frac{u'(x)}{1+u(x)}\, dx$

$= \arctan(x) + \frac{1}{2}\int \frac{1}{1+u}\, du$

$= \arctan(x) + \frac{1}{2}\ln|1+u(x)| + C$

$= \arctan(x) + \frac{1}{2}\ln(1+x^2) + C.$

2. (10 points) Use the technique of "logarithmic differentiation" to compute the derivative of the function $f(x)$ below. Be sure to show your work.

$$f(x) = \frac{x^4 + 3x^2}{\cos^2(x)\sqrt{x}}.$$

$$\ln(f(x)) = \ln\left(\frac{x^4+3x^2}{\cos^2(x)\cdot\sqrt{x}}\right)$$

$$= \ln(x^4+3x^2) - \left(\ln(\cos^2(x)) + \ln(\sqrt{x})\right)$$

$$= \ln(x^4+3x^2) - 2\ln(\cos(x)) - \frac{1}{2}\ln(x)$$

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\left(\ln(x^4+3x^2) - 2\ln(\cos(x)) - \frac{1}{2}\ln(x)\right)$$

$$= \frac{4x^3+6x}{x^4+3x^2} + \frac{2\sin(x)}{\cos(x)} - \frac{1}{2x}$$

$$f'(x) = \left(\frac{x^4+3x^2}{\cos^2(x)\sqrt{x}}\right)\left(\frac{4x^3+6x}{x^4+3x^2} + \frac{2\sin(x)}{\cos(x)} - \frac{1}{2x}\right)$$

3. (24 points) Compute the derivaties of the following functions. Be sure to show your work carefully, using correct notation.

(a) $\ln(\sin(x)\cos(x))$

$$\frac{d}{dx}\ln(\sin(x)\cos(x)) = \frac{1}{\sin(x)\cos(x)} \cdot \frac{d}{dx}\sin(x)\cos(x)$$

$$= \frac{1}{\sin(x)\cos(x)}\left(-\sin(x)\sin(x) + \cos(x)\cos(x)\right)$$

$$= \frac{\cos^2(x) - \sin^2(x)}{\sin(x)\cos(x)}$$

$$\frac{d}{dx}\ln(\sin(x)\cos(x)) = \frac{d}{dx}\ln(\sin(x)) + \ln(\cos(x))$$

$$= \frac{d}{dx}\ln(\sin(x)) + \frac{d}{dx}\ln(\cos(x))$$

$$= \frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)}$$

(b) $5^{\arcsin(x)}$

$$\frac{d}{dx}5^{\arcsin(x)} = \frac{d}{dx}e^{\ln(5)\arcsin(x)}$$

$$= e^{\ln(5)\arcsin(x)} \cdot \frac{d}{dx}\ln(5)\arcsin(x)$$

$$= 5^{\arcsin(x)} \cdot \frac{\ln(5)}{\sqrt{1-x^2}}$$

(c) $\log_3(5x^4 + 7x^2 + 12)$

$$\frac{d}{dx} \log_3 (5x^4 + 7x^2 + 12) = \frac{d}{dx} \frac{\ln(5x^4 + 7x^2 + 12)}{\ln(3)}$$

$$= \frac{1}{\ln(3)} \cdot \frac{d}{dx} \ln(5x^4 + 7x^2 + 12)$$

$$= \frac{1}{\ln(3)} \cdot \left(\frac{20x^3 + 14x}{5x^4 + 7x^2 + 12} \right)$$

(d) $\arctan\left(\sqrt{x}\right)$

$$\frac{d}{dx} \arctan(\sqrt{x}) = \frac{1}{1 + (\sqrt{x})^2} \cdot \frac{d}{dx} \sqrt{x}$$

$$= \left(\frac{1}{1 + x} \right) \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}(1 + x)}$$

4. (10 points) A sum of \$1,000 is deposited in an account which earns interest with **continuous compounding**. After one year, the amount has increased to \$1,200.

(a) How much money will be in the account after 3 years? [This amount can be computed exactly without use of a calculator.]

The hypotheses imply $\frac{A'(t)}{A(t)} = r$ for some relative rate r.

Thus $A(t) = 1000e^{rt}$.

Also $A(1) = 1000e^{r}$ and $A(1) = 1200$.

These imply $1000e^{r} = 1200$

$$e^{r} = \frac{1200}{1000} = 1.2$$

Finally, $A(3) = 1000e^{r\cdot 3} = 1000(1.2)^3 = 1000(1.728)$

$= 1728$

$$\begin{array}{r} 1.2 \\ \times\ 1.2 \\ \hline 24 \\ 12\ \ \\ \hline 1.44 \\ \times\ 1.2 \\ \hline 288 \\ 144\ \ \\ \hline 1.728 \end{array}$$

\$1728

(b) Write down an **exact mathematical expression** for the amount of time it takes for the account to be worth \$10,000.

$$10000 = 1000(1.2)^t$$

$$(1.2)^t = 10$$

$$t\ln(1.2) = \ln(10)$$

$$t = \frac{\ln(10)}{\ln(1.2)}$$

Can also solve for $r = \ln\left(\frac{6}{5}\right)$ and solve $10000 = 1000e^{\ln(\frac{6}{5})t}$

$$\ln(10) = \ln\left(\tfrac{6}{5}\right)t$$

$$t = \ln(10)/\ln(6/5)$$

5. (15 points) Compute the following limits. **Show the steps required to perform each computation.** If you use L'Hôpital's Theorem, show where you use it and explain what type of limit you are using it on (i.e., which indeterminate form).

(a) $\lim_{x\to 0^+} (e^x + 3x)^{1/x}$

Use L'Hôpital's Theorem in type " 1^∞ ".

$$\lim_{x\to 0^+} \log(e^x+3x)^{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\log(e^x+3x)}{x} = \lim_{x\to 0^+} \frac{\left(\frac{1}{e^x+3x}\right)(e^x+3)}{1} = \lim_{x\to 0^+} \frac{e^x+3}{e^x+3x}$$

$$= \frac{\lim_{x\to 0^+} e^x+3}{\lim_{x\to 0^+} e^x+3x} = \frac{4}{1} = 4$$

$$\lim_{x\to 0^+} (e^x+3x)^{1/x} = e^{\lim_{x\to 0^+} \log(e^x+3x)^{\frac{1}{x}}} = e^4$$

(b) $\lim_{x\to\infty} \frac{(\ln(x))^2}{x} = *$

Since $\lim_{x\to\infty} (\ln(x))^2 = \infty$ and $\lim_{x\to\infty} x = \infty$, we can use L'Hôpital's Theorem: Form " $\left[\frac{\infty}{\infty}\right]$ " (Used twice.)

$$* = \lim_{x\to\infty} \frac{2\ln(x)\cdot\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2\ln(x)}{x} = \lim_{x\to\infty} \frac{(2/x)}{1} = \lim_{x\to\infty} \frac{2}{x} = 0$$

(c) $\lim_{x\to 0} \frac{\sin(5x^3)}{\sin(3x^2)} = *$

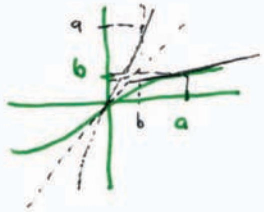
Since $\lim_{x\to 0} \sin(5x^3) = 0$ and $\lim_{x\to 0} \sin(3x^2) = 0$ we can use L'Hôpital's Theorem: Form " $\left[\frac{0}{0}\right]$ "

$$* = \lim_{x\to 0} \frac{15x^2\cos(5x^3)}{6x\cos(3x^2)} = \lim_{x\to 0} \frac{15x\cos(5x^3)}{6\cos(3x^2)} = \frac{\lim_{x\to 0} 15x\cos(5x^3)}{\lim_{x\to 0} 6\cos(3x^2)} = \frac{0}{6} = 0$$

6. (9 points)

(a) Suppose that $f(x)$ is a differentiable function with known inverse $g(x) = f^{-1}(x)$. Assume as well that $f(a) = b$, so that the point (a, b) lies on the graph of $f(x)$. Express the derivative $g'(b) = (f^{-1})'(b)$ in terms of the derivative of f. **[No work need be shown for this question.]**

$g'(b) = \frac{1}{f'(a)}$



(b) Explain (briefly) why the function $f(x) = \sin(x)$ does **not** have an inverse. What is required to define the function $\arcsin(x)$?

sin(x) does not have an inverse because it is not one-to-one.

arcsin(x) is the inverse of sin(x) when restricted to the domain $[-\pi/2, \pi/2]$ on which sin(x) is one-to-one.

(c) Use the fact that $\sin(\pi/4) = 1/\sqrt{2}$ and your formula for (a) and explanation in (b) to find $\arcsin'(1/\sqrt{2})$.

The point $(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$ lies on the graph of sin(x) and $\frac{d}{dx}\sin(x) = \cos(x)$,

$\arcsin'(\frac{1}{\sqrt{2}}) = \frac{1}{\cos(\pi/4)} = \frac{1}{(1/\sqrt{2})} = \sqrt{2}$

7. (8 points) Compute the following integrals.

(a) $\int \arctan(x)\ dx = *$

$u(x) = \arctan(x) \quad v'(x) = 1$

$u'(x) = \frac{1}{1+x^2} \quad v(x) = x$

$* = x \arctan(x) - \int \frac{x}{1+x^2}\, dx$

$= x \arctan(x) - \frac{1}{2} \int \frac{w'(x)}{1+w(x)}\, dx \qquad w(x) = x^2, \quad w'(x) = 2x$

$= x \arctan(x) - \frac{1}{2} \int \frac{1}{1+w}\, dw$

$= x \arctan(x) - \frac{1}{2} \ln(1+x^2) + C$

$* = x \arctan(x) - \frac{1}{2} \int \frac{w'(x)}{w(x)}\, dx \qquad w(x) = 1+x^2, \quad w'(x) = 2x$

$= x \arctan(x) - \frac{1}{2} \int \frac{1}{w}\, dw$

$= x \arctan(x) - \frac{1}{2} \ln(1+x^2) + C$

(b) $\int x^2 e^x\ dx = *$

$u(x) = x^2 \quad v'(x) = e^x$

$u'(x) = 2x \quad v(x) = e^x$

$* = x^2 e^x - \int 2x\, e^x\, dx$

Redefine $u(x)$:

$u(x) = 2x \quad v'(x) = e^x$

$u'(x) = 2 \quad v(x) = e^x$

$= x^2 e^x - \left[2x e^x - \int 2e^x\, dx \right]$

$= x^2 e^x - 2x e^x + 2 \int e^x\, dx$

$= x^2 e^x - 2x e^x + 2e^x + C$