The Huppert conjecture revisited

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- $\mathrm{cd}(G)$ the set of degrees of irreducible characters of G.

An important program in character theory of finite groups is to determine the structure of group G when $\mathrm{cd}(G)$ is known. For example:

Example

- $\mathrm{cd}(G) = \{1\} \Longrightarrow G$ is Abelian.
- $|\mathrm{cd}(G)| \leq 3 \Longrightarrow G$ is a solvable group (Isaacs).

⚠ As $|\mathrm{cd}(A_5)| = 4$, not all finite groups with 4 character degrees are solvable. The structure of non-solvable groups with 4 character degrees were classified by Malle and Moretó (2004).

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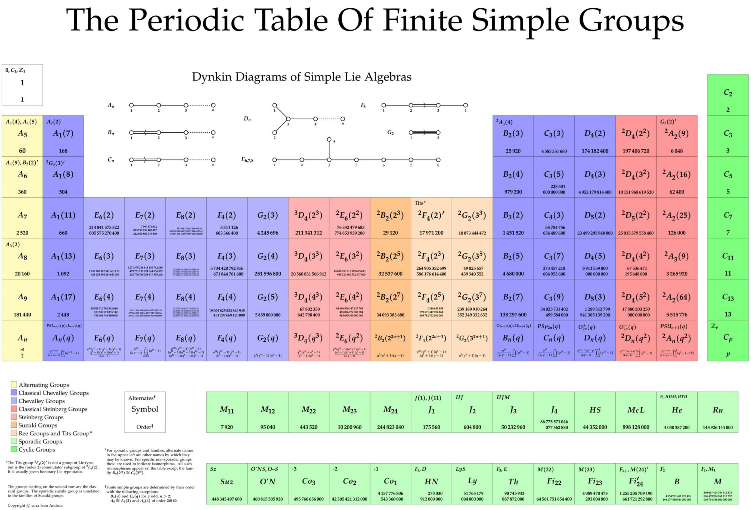
In contrast to finite solvable groups, it seems that “nearly" simple groups have a strong connection with their character degrees. By nearly simple groups we mean almost simple and quasi-simple extensions of non-Abelian simple groups.

Visualized CFSG



The Huppert Conjecture

Huppert's conjecture

A celebrated conjecture of Huppert states that non-Abelian simple groups are determined up to an Abelian direct factor by the set of their character degrees.

Conjecture (Huppert, 2000)

Let H be a non-Abelian simple group and G be a finite group such that $\mathrm{cd}(G) = \mathrm{cd}(H)$. Then $G \cong H \times A$ for some Abelian group A.

Huppert's conjecture

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- Step 5: Showing that $G/C_G(G') \cong H$.

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- Sporadic simple groups by various authors.

Huppert's conjecture for $\mathrm{PSL}_5(q)$: a progress report

Using a modified version of the method posed by Huppert, we have been working to confirm this conjecture for the family of projective special linear groups $\mathrm{PSL}_5(q)$ with $q \geq 11$.

Indeed, we have verified Steps 1, 2, 4, and 5, and the verification of Step 3 is in progress.

This will be the first time that the Huppert conjecture is verified for a simple Lie-type group of rank 4.

Almost simple groups of Lie type

Based on the progress of Huppert's conjecture, we are interested in extending this conjecture from non-Abelian simple groups to almost simple groups.

Definition

A group G is called an **almost simple** group (with socle S) if there exists a non-Abelian simple group S such that $S \leq G \leq \mathrm{Aut}(S)$.

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- By almost simple groups of Lie type we mean the almost simple groups whose socle is a simple Lie-type group.
- By CFSG, most of (almost) simple groups are of Lie type.

A variation of Huppert's conjecture for almost simple groups

Conjecture (Conjecture A)

Let H be an almost simple group of Lie type and G be a finite group such that $\mathrm{cd}(G) = \mathrm{cd}(H)$. Then there exists an Abelian normal subgroup A of G such that $G/A \cong H$.

- Shirjian, F and Iranmanesh A. (2020). Extending Huppert's conjecture to almost simple groups of Lie type. Illinois J. Math. 64(1), 49–69.

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- The case in which H is an almost simple "sporadic" group has been already worked out by Alavi-Daneshkhah-Jafari (2017).

A variation of Huppert's conjecture for almost simple groups

Example

Let $H = \mathrm{PGL}_3(q)$ and $G = \mathrm{GL}_3(q)$ with $q > 4$ and $q \equiv 1 \pmod 3$. Then

$$\mathrm{cd}(G) = \mathrm{cd}(H),$$

but $\mathrm{GL}_3(q)$ is not a (semi-)direct product of its center and $\mathrm{PGL}_3(q)$.

The example shows that, despite of Huppert's conjecture, the group G in Conjecture A is not necessarily a semi-direct product of H and A.

A variation of Huppert's conjecture for almost simple groups

Example

Let $G = (C_2 \times C_2) : \mathrm{PGL}_3(q)$, a semidirect product of $C_2 \times C_2$ and $\mathrm{PGL}_3(q)$ in which the action of $\mathrm{PGL}_3(q)$ on $C_2 \times C_2$ is given by $\varphi : \mathrm{PGL}_3(q) \to \mathrm{GL}_2(2)$,

$$\varphi(x) = \begin{cases} \mathrm{id} & x \in \mathrm{PSL}_3(q) \\ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} & x = \delta, \end{cases}$$

where $q \equiv 1 \pmod 3$, and δ is a diagonal automorphism of $\mathrm{PSL}_3(q)$ of order 3. Then $3 \in \mathrm{cd}(G) \setminus \mathrm{cd}(\mathrm{PGL}_3(q))$.

Complex group algebras of groups of Lie type

Brauer's program on studying complex group algebras

For G being a finite group, a general question here is

What $\mathbb{C}G$ knows about the structure of G?

In other words, assuming G to have an algebraic property $\mathscr{P}$ and $\mathbb{C}G \cong \mathbb{C}H$, does it imply that H has also the property $\mathscr{P}$?

Brauer's program on studying complex group algebras

Brauer's Question

When two finite groups have isomorphic (complex) group algebras?

Definition

A group G is said to be uniquely determined by the structure of its complex group algebra if for any group H, the isomorphism $\mathbb{C}G \cong \mathbb{C}H$ of algebras implies the isomorphism $G \cong H$ of the base groups.

Of course not all the finite groups are uniquely determined by the structure of their complex group algebras.

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Conjecture B

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So far Conjecture A has been verified:

- for symmetric groups (Tong-Viet, 2011)
- for almost simple groups with Sporadic socle (Alavi et. al., 2017)
- for almost simple groups with socle $\mathrm{PSL}_2(q)$ (Ahanjideh et. al., 2017).

Complex group algebras of almost simple groups

In the following, $\epsilon = +$ if G is of linear type, and $\epsilon = -$ if G is of unitary type.

Theorem (Sh-Iranmanesh-Shafiei, 2018-2020)

Let $n \geq 2$ and $PSL_n^{\epsilon}(q) \leq G \leq PGL_n^{\epsilon}(q)$ where $q - \epsilon 1$ is not a divisor of n or $n-1$. Then G is determined up to isomorphism by the structure of its complex group algebra.

• Shirjian, F and Iranmanesh A. (2018). Complex group algebras of almost simple groups with socle $\mathrm{PSL}_n(q)$. Comm. Algebra 46(2) 552–573.
• Shirjian, F., Iranmanesh, A., and Shafiei, F. (2020). Complex group algebras of almost simple unitary groups, Comm. Algebra 48(5), 1919–1940.

Thank you for your attention!