Punctured groups for exotic fusion systems

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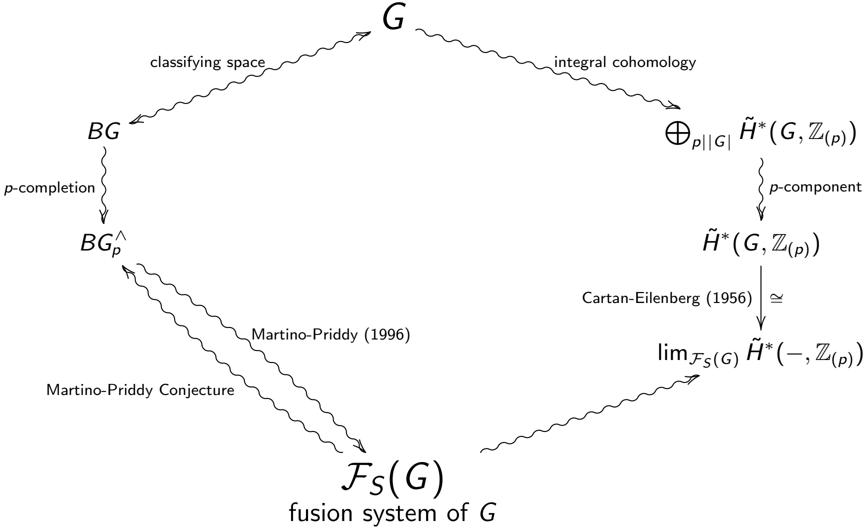
Invariants of a finite group around a prime p

G a finite group, S a Sylow p-subgroup of G



Fusion systems

Fusion systems encode conjugation maps

G a finite group, p a prime, S a Sylow p-subgroup of G

Fusion system of a finite group

$\mathcal{F} = \mathcal{F}_S(G)$, a category:

- objects: subgroups $P \leq S$;
- morphisms: conjugation homomorphisms induced from G

$$\mathrm{Hom}_{\mathcal{F}}(P, Q) := \mathrm{Hom}_G(P, Q) = \{c_g : P \hookrightarrow Q \mid g \in G, gPg^{-1} \leq Q\}$$

- $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Hom}_{\mathcal{F}}(P, P) = N_G(P)/C_G(P)$

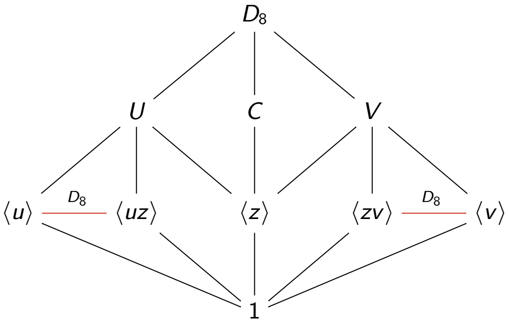
Saturated fusion systems (Puig)

Let S be a finite p-group.

- A fusion system on S is a category $\mathcal{F}$ with objects $\{P \leq S\}$ and morphisms satisfying $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ and one more weak axiom.
- The fusion system $\mathcal{F}$ is saturated if $\mathrm{Inn}(S) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(S))$ and one more axiom holds (which allows you to extend certain morphisms to larger subgroups).

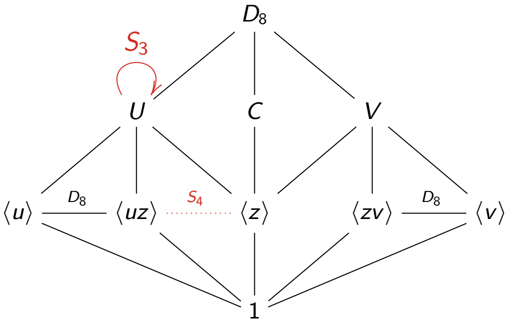
Fusion systems with $S \cong D_8$

$\mathcal{F}_{D_8}(D_8)$



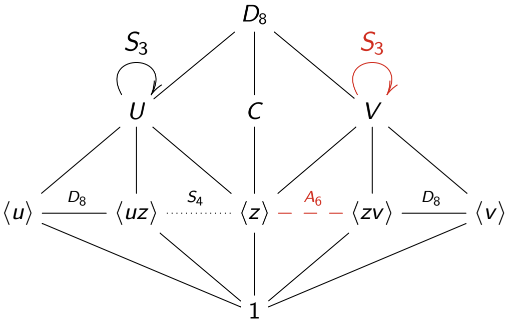
Fusion systems with $S \cong D_8$

$\mathcal{F}_{D_8}(S_4)$



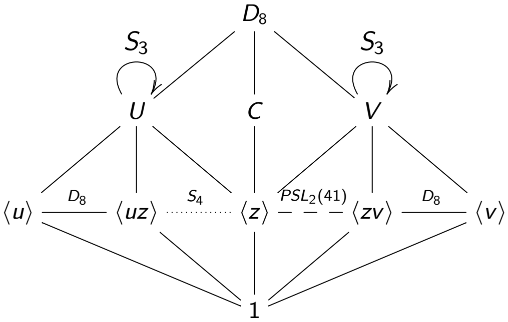
Fusion systems with $S \cong D_8$

$\mathcal{F}_{D_8}(A_6)$



Fusion systems with $S \cong D_8$

$\mathcal{F}_{D_8}(PSL_2(q)), q \equiv \pm 9 \pmod{16}$



Exotic fusion systems

Exotic fusion system

Not realizable, i.e. not of the form $\mathcal{F}_S(G)$ for any finite group G with Sylow S.

The Benson-Solomon systems at $p = 2$

$\mathrm{Sol}(q)$ on S, a Sylow 2-subgroup of $\mathrm{Syl}_2(\mathrm{Spin}_7(q))$, q odd.

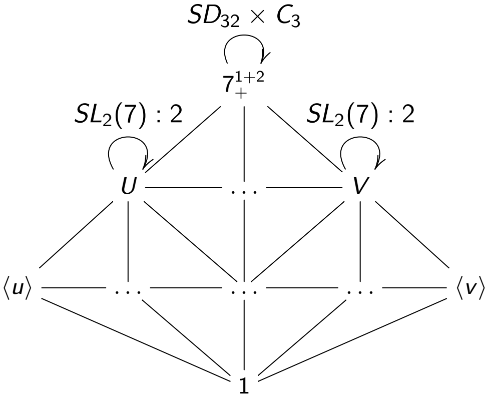
- $|Z(S)| = 2$ and $N_{\mathrm{Sol}(q)}(Z(S)) \cong \mathcal{F}_S(\mathrm{Spin}_7(q))$ (normalizer subsystem).
- $\mathrm{Sol}(q)$ has one conjugacy class of involutions; $\mathrm{Spin}_7(q)$ has two.
- In $H = \mathrm{Spin}_7(q)$, maximal torus T of rank 3 with $N_H(T)/T = C_2 \times S_4$; in $\mathrm{Sol}(q)$ maximal 2-torus $T_2 \leq S$ has $\mathrm{Aut}_{\mathrm{Sol}(q)}(T_2) = C_2 \times GL_3(2)$.

At odd primes p: lots of them

- Ruiz-Viruel (2004): Three at $p = 7$ on $S = 7^{1+2}_+$.
- Oliver (2014), Craven-Oliver-Semeraro (2017), Oliver-Ruiz (2020): Fusion systems on S having abelian A with $|S : A| = p$.
- Parker-Stroth (2015): Fusion systems on $S = p^{1+2n}_+ \rtimes C_p$.
- ...

Picture of a Ruiz-Viruel exotic fusion system

$\mathcal{F} = RV_3$ on $S = 7_+^{1+2}$: all eight subgroups $C_7 \times C_7$ of order 7^2 conjugate, all subgroups of order 7 conjugate, $\mathrm{Aut}_{\mathcal{F}}(7^2) \cong SL_2(7) : 2$.



Transporter systems

Transporter categories encode group elements doing the conjugating

- Δ a (nonempty) "overgroup-closed", "G-conjugacy invariant" collection of subgroups of S

Transporter category of a group: $\mathcal{T}_\Delta(G)$

objects: $P \in \Delta$
morphisms: $N_G(P, Q) = \{g \in G \mid gPg^{-1} \leq Q\}$ (composition: mult. in G)

Note: Have quotient functor $\pi \colon \mathcal{T}_\Delta(G) \to \mathcal{F}_S(G)$, $g \mapsto c_g$, which is the inclusion on objects and surjective on morphisms. Say $\mathcal{T} = \mathcal{T}_\Delta(G)$ is associated with $\mathcal{F} = \mathcal{F}_S(G)$.

Examples

- $\Delta = \{S\}$: $\mathcal{T}_\Delta(G)$ is essentially $N_G(S)$.
- $\Delta =$ all nonidentity subgroups of S: the full p-local structure of G. Write as $\mathcal{T}_S^*(G)$, and call it the **punctured group** for G.
- $\Delta =$ all subgroups of S: $\mathcal{T}_S(G) := \mathcal{T}_\Delta(G)$ is essentially G, because $G = N_G(1)$.

As Δ gets larger, $\mathcal{T}_\Delta(G)$ interpolates between $N_G(S)$ and G.

Transporter systems and linking systems

Transporter system associated with a saturated fusion system (Oliver-Ventura)

Category $\mathcal{T}$ with object set Δ and functors

$$\mathcal{T}_\Delta(S) \xrightarrow{\iota} \mathcal{T} \xrightarrow{\pi} \mathcal{F}$$

satisfying axioms that model the properties of $\mathcal{T}_\Delta(G)$.

p-centric subgroup

A p-subgroup $Q \leq S$ such that $C_G(Q) = Z(Q) \times O_{p'}(C_G(Q))$. Write $\mathcal{F}_S(G)^c$ or $\mathcal{F}^c$ for the set of centric subgroups of S.

Examples of transporter systems

- $\mathcal{T}_\Delta(G)$, any G, any nonempty Δ.
- The centric linking system $\mathcal{L} = \mathcal{L}_S^c(G)$ with $\Delta = \mathcal{F}_S(G)^c$: Here,

$$\mathrm{Mor}_{\mathcal{L}}(P, Q) = N_G(P, Q)/O_{p'}(C_G(P)).$$

Additional motivation for transporter systems: Centric linking systems and the Martino-Priddy Conjecture

Martino-Priddy Conjecture

For two finite groups G and H,

$$BG_p^\wedge \simeq BH_p^\wedge \iff \mathcal{F}_p(G) \cong \mathcal{F}_p(H)$$

- ($\Longrightarrow$) Martino-Priddy (1996)
- ($\Longleftarrow$) Oliver (2004: p odd; 2006: $p = 2$)

Broto-Levi-Oliver (2003): $\mathcal{L}_S^c(G)$ recovers $BG_p^\wedge$

$$|\mathcal{L}_S^c(G)|_p^\wedge \simeq BG_p^\wedge$$

Thus, the "if" direction of the Martino-Priddy conjecture is equivalent to the uniqueness of $\mathcal{L}_S^c(G)$ (up to isomorphism of transporter systems).

Punctured groups

Punctured groups

Definition

A punctured group for the saturated fusion system $\mathcal{F}$ is a transporter system $\mathcal{T}$ with object set Δ all nonidentity subgroups of S.

Example

$\mathcal{T}_S^*(G)$ (for a finite group G) is a punctured group for $\mathcal{F}_S(G)$.

Question (Chermak)

Which exotic fusion systems have punctured groups?

Motivation

1. *Aesthetics: how close is an exotic system to a group?*
2. *Usually, $\pi_1(\mathcal{T}_S^*(G))$ is a much better approximation to G than is $\pi_1(\mathcal{L}_S^c(G))$.*
3. *It is not known whether every block fusion system $\mathcal{F}_D(kGb)$ is realizable by a finite group. Is each block fusion system nevertheless realizable by a punctured group?*

Approximating a (nonexistent) group by transporter categories

Given

- a(n) (exotic, say) saturated fusion system $\mathcal{F}$ over S, and
- a reasonable filtration of the poset of subgroups of S:

$$\{S\} = \Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_n \subset \cdots \subset \Delta_N = \mathrm{Sub}^*(S) \subset \Delta_{N+1} = \mathrm{Sub}(S),$$

Try to build inductively

$$\mathcal{T}_0 \hookrightarrow \mathcal{T}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{T}_n \hookrightarrow \cdots \hookrightarrow \mathcal{T}_N \hookrightarrow \mathcal{T}_{N+1}$$

associated with $\mathcal{F}$.

- Okay for $\Delta_0 = \{S\}$.
  - The group extension problem $1 \to Z(S) \to N_{\mathcal{T}_0}(S) \to \mathrm{Aut}_{\mathcal{F}}(S) \to 1$ is solvable.
- More generally, for a centric subgroup P, the group extension problem $1 \to Z(P) \to N_{\mathcal{T}_k}(P) \to \mathrm{Aut}_{\mathcal{F}}(P) \to 1$ is solvable.

Theorem (**Chermak** (2013), Oliver, Glauberman-L.)

Can build $\mathcal{T}_n$ up to $\Delta_n = \mathcal{F}^c$ for any saturated fusion system. Get analogue $\mathcal{L} = \mathcal{T}_n$ of the centric linking system of a group.

Punctured groups for some exotic fusion systems

A necessary condition and a sufficient condition for the existence of a punctured group

Normalizer subsystem $N_{\mathcal{F}}(X)$ for $X \leq S$

objects: $P \leq N_S(X)$;
morphisms: $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ having an extension $\tilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(XP, XQ)$ such that $\tilde{\varphi}(X) = X$.

The largest normal p-subgroup of a fusion system

$O_p(\mathcal{F})$ is the largest subgroup X of S such that $\mathcal{F} = N_{\mathcal{F}}(X)$.

Observation (necessary condition for a punctured group)

If $\mathcal{F}$ has a punctured group $\mathcal{T}$, then the normalizer $N_{\mathcal{F}}(X)$ of each $1 \neq X \leq S$ is not exotic. (Because $N_{\mathcal{F}}(X) = \mathcal{F}_{N_S(X)}(N_{\mathcal{T}}(X))$.)

Theorem (Henke, sufficient condition)

If $\mathcal{F}$ is of "characteristic p-type" (i.e. $O_p(N_{\mathcal{F}}(X)) \in \mathcal{F}^c$ for each $1 \neq X \leq S$), then $\mathcal{F}$ has a punctured group.

Survey of some exotic fusion systems at odd primes

Has a punctured group?

- **Ruiz-Viruel systems (2004)** over 7_+^{1+2}: Yes.
- **Oliver's exotic systems (2014)** (A abelian, $|S : A| = p$): Roughly half do, half do not.
- **Clelland-Parker systems (2010)**: Roughly half do, half don't.
- **Parker-Stroth systems (2015)** over $S \cong p_+^{1+2n} \rtimes C_p$: Yes.

A punctured group for $\mathrm{Sol}(q)$

- In $\mathcal{F} = \mathrm{Sol}(q)$ at the prime 2, each $N_{\mathcal{F}}(X)$ with $1 \neq X \leq S$ is realizable by a finite group. For example, recall

$$N_{\mathrm{Sol}(q)}(Z(S)) \cong \mathcal{F}_S(\mathrm{Spin}_7(q)).$$

- $\{\mathrm{Sol}(3^{2^k}) \mid k \geq 0\}$ is a nonredundant list of $\mathrm{Sol}(q)$'s.

Theorem (Solomon, 1974), rephrased

$\mathrm{Sol}(3^{2^k})$ *is exotic, i.e. it has no associated transporter system with objects all subgroups of* S*.*

Theorem (Henke-Libman-L., 2020)

$\mathrm{Sol}(3^{2^k})$ *has a punctured group, i.e. a transporter category on nonidentity subgroups of* S*, if and only if* $k = 0$*.*

- For $k = 0$, can build one with $C_{\mathcal{T}}(Z(S)) = \mathrm{Spin}_7(3)$.
- Others with $k = 0$ might exist with $C_{\mathcal{T}}(Z(S)) = \mathrm{Spin}_7(3^{1+6a})$ for certain $a \neq 0$, but we can't prove or disprove.
- Idea for $k > 0$: Can show $C_{\mathcal{T}}(Z(S)) \sim \mathrm{Spin}_7(q)$ for some odd q. Look at two different maximal tori: $T \cong C_{q-1}^3$ and $T' \cong C_{q+1}^3$. Get $GL_3(2)$ must have faithful action on $\mathbb{F}_p^3$ for every prime p dividing $(q-1)(q+1)$.
- Uses a "Signalizer functor theorem for punctured groups".

Application to the topology of classifying spaces

An application: punctured groups and the subgroup decomposition

- The *centric orbit category* $\mathcal{O}(\mathcal{F}^c)$ has objects the $\mathcal{F}$-centric subgroups, and morphisms $\mathrm{Inn}(Q)\backslash\mathrm{Hom}_{\mathcal{F}}(P, Q)$.
- Subgroup decomposition: the functor $B\colon \mathcal{O}(\mathcal{F}^c) \to \mathrm{hoTop}$ given by $P \mapsto BP$ is liftable to a unique functor $\tilde{B}\colon \mathcal{O}(\mathcal{F}^c) \to \mathrm{Top}$ (up to htpy equivalence), and
$$|\mathcal{L}| \simeq \mathrm{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \tilde{B},$$
where $\mathcal{L}$ is the centric linking system of $\mathcal{F}$.
- Bousfield-Kan spectral sequence for $H^i(|\mathcal{L}|, \mathbf{F}_p)$: $E_2 = \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^i H^j(-, \mathbf{F}_p)$.

Theorem (HLL)

If $\mathcal{F}$ has a punctured group, then the cohomology functors $H^j(-, \mathbf{F}_p)$ over $\mathcal{O}(\mathcal{F}^c)$ have vanishing higher limits:

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^i H^j(-, \mathbf{F}_p) = 0$$

for all $i \geq 1$.

- Dwyer for $\mathcal{F}_S(G)$, additional work by Díaz-Park for certain exotic $\mathcal{F}$.
- With Theorem, more direct proof that $H^i(|\mathcal{L}|, \mathbf{F}_p)$ is computable by stable elements (Broto-Levi-Oliver).

Thank you