Problem 1.
a) Is there a one-to one and onto function $f : (0,1) \longrightarrow (0,1)$ such that $f' = f^{-1}$, i.e. the derivative of f equals the inverse of f?

b) Is there a one-to one and onto function $f : (0,\infty) \longrightarrow (0,\infty)$ such that $f' = f^{-1}$, i.e. the derivative of f equals the inverse of f?

Solution. a) Suppose that such f exists. Since $f' = f^{-1}$ and f^{-1} assumes only positive values, the function f is strictly increasing (exercise: show that if $g : (a,b) \longrightarrow \mathbb{R}$ is continuous and one-to-one then g is either strictly increasing or strictly decreasing). It follows that $\lim_{x\to 0^+} f(x) = u$ and $\lim_{x\to 1^-} f(x) = w$ exist. Note that f assumes values in the interval (u,w). Since f is onto $(0,1)$ we must have $u = 0$ and $w = 1$. Defining $f(0) = 0$ and $f(1) = 1$ extends f to a continuous bijection $f : [0,1] \longrightarrow [0,1]$. Thus f becomes a continuous function on the closed interval $[0,1]$ and differentiable on $(0,1)$. We may now apply the Mean Value Theorem, which says that

$$1 = \frac{f(1) - f(0)}{1 - 0} = f'(u)$$

for some $u \in (0,1)$. On the other hand, $f'(u) = f^{-1}(u) \in (0,1)$, which contradicts the equality $f'(u) = 1$. This shows that f can not exist.

b) Here the answer is actually positive. When looking for a function with some particular properties, a good first step is to look at some common functions. We ask whether f can be of the form $f(x) = ax^t$ for some positive a and t (as in part a), it is easy to see that f is increasing and positive). We have

$$f^{-1}(x) = a^{\frac{-1}{t}} x^{\frac{1}{t}} \quad \text{and} \quad f'(x) = atx^{t-1}.$$

The equality $f' = f^{-1}$ will hold if

$$\frac{1}{t} = t - 1 \quad and \quad at = a^{\frac{-1}{t}}.$$

The first equality says that $t^2 - t - 1 = 0$. This equation has only one positive root $t = (1+\sqrt{5})/2$. The second equality says that

$$t = a^{\frac{-1}{t}-1} = a^{\frac{-(1+t)}{t}} = a^{\frac{-t^2}{t}} = a^{-t}$$

(we used the fact that $1 + t = t^2$). Note that $-1/t = 1 - t$, so

$$a = t^{-1/t} = t^{1-t}.$$

In summary, the function $f(x) = t^{1-t}x^t$, where $t = (1+\sqrt{5})/2$, satisfies the conditions of part b).

Question. Problem 1 leads to the following problems, to which we do not know a solution, and which could constitute a nice undergraduate research project.

(A) Find all intervals (a,b) for which there is a bijective and differentiable function $f : (a,b) \longrightarrow (a,b)$ such that $f'(x) = f^{-1}(x)$ for all $x \in (a,b)$

(B) For each interval found in (A), describe all possible functions f.

At present, we do not know if the function f in our solution to part b) is the only functions satisfying the requirements. Note that $f(t) = t$, so the intervals

$$(0, (1+\sqrt{5})/2), \qquad ((1+\sqrt{5})/2, \infty), \qquad (0, \infty)$$

are examples of intervals in problem (A). We do not know any other examples.