**Problem 3.** Is the number $1^6 + 2^6 + 3^6 + \ldots + 999999^6$ divisible by $10^6$?

**Solution.** We will show that the number $M = 1^6 + 2^6 + 3^6 + \ldots + 999999^6$ is not divisible by $2^6$. We will need the following lemma.

**Lemma.** Let $n > 0$ be even. Then for every $k \geq 1$ we have

$$\sum_{i=1}^{2^k-1} i^n \equiv 2^{k-1} \pmod{2^k} .$$

We prove the lemma by induction on $k$. For $k = 1$ we have

$$\sum_{i=1}^{2-1} i^n = 1 \equiv 2^{1-1} \pmod 2$$

so the lemma is true for $k = 1$. Suppose now that the lemma is true for some $k \geq 1$. Thus

$$\sum_{i=1}^{2^k-1} i^n \equiv 2^{k-1} \pmod{2^k} \text{ and } 2\sum_{i=1}^{2^k-1} i^n \equiv 2 \cdot 2^{k-1} = 2^k \pmod{2^{k+1}} .$$

We have

$$\sum_{i=1}^{2^{k+1}-1} i^n = \sum_{i=1}^{2^k-1} i^n + (2^k)^n + \sum_{i=1}^{2^k-1} (2^{k+1} - i)^n \equiv \sum_{i=1}^{2^k-1} i^n + \sum_{i=1}^{2^k-1} (-i)^n = 2\sum_{i=1}^{2^k-1} i^n \equiv 2^k \pmod{2^{k+1}}$$

(in the last equality we used the assumption that $n > 0$ is even). Thus the lemma is true for $k+1$. This completes our proof of the lemma.

We are now ready to show that $M$ is not divisible by $2^6$. Note that

$$M = \sum_{k=0}^{5^6-1} \sum_{i=1}^{2^6} (2^6 k + i)^6.$$

Now, for every $k$ we have

$$\sum_{i=1}^{2^6} (2^6 k + i)^6 \equiv \sum_{i=1}^{2^6-1} i^6 \pmod{2^6} .$$

Thus, using our lemma, we get

$$M \equiv 5^6 \sum_{i=1}^{2^6-1} i^6 \equiv 5^6 \cdot 2^5 \not\equiv 0 \pmod{2^6} .$$

Thus $M$ is not divisible by $2^6$.

**Exercise.** Is $M$ divisible by $5^6$?

**Exercise.** Investigate for which $m, n$ the number $1^n + 2^n + \ldots + (m-1)^n$ is divisible by $m$.