Problem 6. For which polynomials $f(x)$ the limit

$$\lim_{x\to\infty}\left(\sqrt[1013]{f(x+2)} - 2\sqrt[1013]{f(x+1)} + \sqrt[1013]{f(x)}\right)$$

is finite and non-zero?

Solution. Our key tool is the Mean Value Theorem:

Mean Value Theorem. *If f is a function continuous on the closed interval $[a,b]$ and differentiable on (a,b) then*

$$\frac{f(b)-f(a)}{b-a} = f'(u) \text{ for some } u \in (a,b).$$

We will also need the folowing simple fact: if $p(x) = sx^n + \textit{lower terms}$ and $q(x) = tx^n + \textit{lower terms}$ are two polynomials of the same degree n with leading coefficients s and t respectively (so $s \neq 0 \neq t$) then

$$\lim_{x\to\infty}\frac{p(x)}{q(x)} = \frac{s}{t}.$$

We are ready to start our solution. Let $n > 0$ be an integer and let $f(x) = ax^m + \textit{lower terms}$ be a polynomial of degree m with the leading coeaddicient $a \neq 0$. In the case when n is even we will assume that $a > 0$. Define the function $F(x)$ as follows:

$$F(x) = \sqrt[n]{f(x)} = f(x)^{1/n}.$$

Then there is B such that $F(x)$ is well defined and has derivatives of all orders for $x > B$. Let $G(x) = F(x+1) - F(x)$. Then

$$G(x+1) - G(x) = F(x+2) - 2F(x+1) + F(x).$$

By the mean value theorem we have

$$G(x+1) - G(x) = G'(u_x) = F'(u_x+1) - F'(u_x) \text{ fors some } u_x \in (x, x+1).$$

Again by the mean value theorem, we have

$$F'(u_x+1) - F'(u_x) = F''(w_x) \text{ fors some } w_x \in (u_x, u_x+1).$$

Combining these two observations, we see that

$$F(x+2) - 2F(x+1) + F(x) = F''(w_x) \text{ fors some } w_x \in (x, x+2).$$

It follows that if $L = \lim_{y\to\infty} F''(y)$ exists then

$$\lim_{x\to\infty}\left(F(x+2) - 2F(x+1) + F(x)\right) = L.$$

Now we have

$$F'(x) = \frac{1}{n} f(x)^{(1-n)/n} f'(x)$$

and

$$F''(x) = \frac{1-n}{n^2} f(x)^{(1-2n)/n} f'(x)^2 + \frac{1}{n} f(x)^{(1-n)/n} f''(x) = \frac{1}{n}\left(\frac{f''(x)^n}{f(x)^{n-1}}\right)^{1/n} - \frac{n-1}{n^2}\left(\frac{f'(x)^{2n}}{f(x)^{2n-1}}\right)^{1/n}.$$

Note that $x^{2n} f''(x)^n$ is a polynomial of degree mn with leading coefficient $a^n m^n (m-1)^n$ and $x^m f(x)^{n-1}$ is a polynomial of degree mn with leading coefficient a^{n-1}. It follows that

$$\lim_{x\to\infty} x^{2n-m}\frac{f''(x)^n}{f(x)^{n-1}} \lim_{x\to\infty}\frac{x^{2n} f''(x)^n}{x^m f(x)^{n-1}} = \frac{a^n m^n (m-1)^n}{a^{n-1}} = am^n(m-1)^n.$$

Taking n-th roots, we get

$$\lim_{x\to\infty} x^{2-m/n}\left(\frac{f''(x)^n}{f(x)^{n-1}}\right)^{1/n} = a^{1/n}m(m-1).$$

Similarly, $x^{2n}f'(x)^{2n}$ is a polynomial of degree $2mn$ with leading coefficient $a^{2n}m^{2n}$ and $x^m f(x)^{2n-1}$ is a polynomial of degree $2mn$ with leading coefficient a^{2n-1}. Therefore

$$\lim_{x\to\infty} x^{2n-m}\frac{f'(x)^{2n}}{f(x)^{2n-1}} = \lim_{x\to\infty}\frac{x^{2n}f'(x)^{2n}}{x^m f(x)^{2n-1}} = \frac{a^{2n}m^{2n}}{a^{2n-1}} = am^{2n}.$$

Taking n-th roots, we get

$$\lim_{x\to\infty} x^{2-m/n}\left(\frac{f'(x)^{2n}}{f(x)^{2n-1}}\right)^{1/n} = a^{1/n}m^2.$$

These computations and our formula for $F''(x)$ yield

$$\lim_{x\to\infty} x^{2-m/n}F''(x) = \frac{1}{n}a^{1/n}m(m-1)-\frac{n-1}{n^2}a^{1/n}m^2 = \frac{a^{1/n}m}{n^2}\left(n(m-1)-(n-1)m\right) = \frac{a^{1/n}m(m-n)}{n^2}.$$

If $m < 2n$ then $2 - m/n > 0$, so we conclude that $\lim_{x\to\infty} F''(x) = 0$ in this case. For $m = 2n$ we get

$$\lim_{x\to\infty} F''(x) = 2a^{1/n}.$$

If $m > 2n$ then $2 - m/n < 0$, so we conclude that $\lim_{x\to\infty} F''(x) = \infty$. We see that

$$\lim_{x\to\infty}\left(F(x+2) - 2F(x+1) + F(x)\right)$$

is finite and non-zero if and only if $m = 2n$.

Problem. a) Let $f(x)$ be a polynomial of degree at most $2n$ such that for every posoitiove integer k the value $f(k)$ is an n-th power of an integer. Prove that $f(x)$ is an n-th power of some polynomial.

b) Do the same as in a) only assuming that $f(k)$ is an n-th power of a rational number.

Remark. It is true in general that if $f(x)$ is a polynomial such that for every posoitiove integer k the value $f(k)$ is an n-th power of an integer then $f(x)$ is an n-th power of some polynomial.