Problem 6. Prove that if a, b, c are positive numbers such that $abc = 1$ then

$$\frac{1}{\sqrt{1+2024a}} + \frac{1}{\sqrt{1+2024b}} + \frac{1}{\sqrt{1+2024c}} \geq \frac{1}{15}.$$

Solution. The only fact needed for our first solution is the celebrated AM-GM inequality.

AM-GM Inequality. *If $a_1, \ldots, a_n$ are non-negative numbers then*

$$\frac{a_1 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

and equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

The quantity on the left in the AM-GM inequality is called the arithmetic mean (AM) of $a_1, \ldots, a_n$ and the quantity on the right is called the geometric mean (GM). Applying the AM-GM inequality to $1/a_1, \ldots, 1/a_n$ yields

$$\frac{\frac{1}{a_1} + \ldots + \frac{1}{a_n}}{n} \geq \frac{1}{\sqrt[n]{a_1 a_2 \ldots a_n}}$$

which leads to the inequality

$$\sqrt[n]{a_1 a_2 \ldots a_n} \geq \frac{n}{\frac{1}{a_1} + \ldots + \frac{1}{a_n}}$$

for all positive numbers $a_1, \ldots, a_n$. The quantity on the right is called the harmonic mean of $a_1, \ldots, a_n$. In particular, the arithmetic mean is always greater or equal than the harmonic mean, with equality if and only if $a_1 = a_2 = \ldots = a_n$. We will need the following special case:

$$x + y + z \geq \frac{9}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}} \tag{1}$$

for any positive real numbers x, y, z.

We are ready to start our first solution. It is based on the following observation.

Lemma 1. *For any positive real numbers u, w we have*

$$\sqrt{1 + 2024uw} \leq 1 + 22u^{\frac{46}{45}} + 22w^{\frac{46}{45}} \tag{2}$$

and the equality holds if and only if $u = w = 1$.

To prove Lemma 1 note that

$$\left(1 + 22u^{\frac{46}{45}} + 22w^{\frac{46}{45}}\right)^2 - 1 = 22\left(u^{\frac{46}{45}} + w^{\frac{46}{45}}\right)\left(2 + 22u^{\frac{46}{45}} + 22w^{\frac{46}{45}}\right).$$

By the AM-GM inequality with $n = 2$ we have

$$u^{\frac{46}{45}} + w^{\frac{46}{45}} \geq 2u^{\frac{23}{45}} w^{\frac{23}{45}}$$

with equality if and only if $u = w$. Now we consider the quantity

$$2 + 22u^{\frac{46}{45}} + 22w^{\frac{46}{45}}$$

as the sum of 46 numbers, two of which are equal to 1, 22 of which are equal to $u^{46/45}$, and 22 of which are equal to $w^{46/45}$, and apply the AM-GM inequality to get

$$2 + 22u^{\frac{46}{45}} + 22w^{\frac{46}{45}} \geq 46\sqrt[46]{1^2 \cdot \left(u^{46/45}\right)^{22}\left(w^{46/45}\right)^{22}} = 46u^{22/45}w^{22/45}$$

with equality if and only if $u = w = 1$. It follows that

$$\left(1 + 22u^{\frac{46}{45}} + 22w^{\frac{46}{45}}\right)^2 - 1 \geq 22 \cdot 2u^{\frac{23}{45}} w^{\frac{23}{45}} \cdot 46u^{\frac{22}{45}} w^{\frac{22}{45}} = 2024uw$$

i.e.
$$1+22u^{\frac{46}{45}}+22w^{\frac{46}{45}}\geq\sqrt{1+2024uw}$$
with equality if and only if $u=w=1$. This completes the proof of Lemma 1.

Since $abc=1$, we have
$$a=\sqrt[3]{\frac{a}{b}}\sqrt[3]{\frac{a}{c}}.$$
Taking $u=\sqrt[3]{\frac{a}{b}}, w=\sqrt[3]{\frac{a}{c}}$ in Lemma 1, we get
$$\sqrt{1+2024a}\leq 1+22\left(\sqrt[3]{\frac{a}{b}}\right)^{\frac{46}{45}}+22\left(\sqrt[3]{\frac{a}{c}}\right)^{\frac{46}{45}}=\sqrt[3]{a}^{\frac{46}{45}}\left(\left(\sqrt[3]{\frac{1}{a}}\right)^{\frac{46}{45}}+22\left(\sqrt[3]{\frac{1}{b}}\right)^{\frac{46}{45}}+22\left(\sqrt[3]{\frac{1}{c}}\right)^{\frac{46}{45}}\right).$$
Setting
$$A=\left(\sqrt[3]{\frac{1}{a}}\right)^{\frac{46}{45}},\ B=\left(\sqrt[3]{\frac{1}{b}}\right)^{\frac{46}{45}},\ C=\left(\sqrt[3]{\frac{1}{c}}\right)^{\frac{46}{45}}$$
the last inequality takes the following form
$$\frac{1}{\sqrt{1+2024a}}\geq\frac{A}{A+22B+22C}$$
and equality holds if and only if $a=b=c=1$.
In exactly the same way we show that
$$\frac{1}{\sqrt{1+2024b}}\geq\frac{B}{B+22A+22C}\quad\text{and}\quad\frac{1}{\sqrt{1+2024c}}\geq\frac{C}{C+22A+22B}.$$
Adding these inequalities, we get
$$\frac{1}{\sqrt{1+2024a}}+\frac{1}{\sqrt{1+2024b}}+\frac{1}{\sqrt{1+2024c}}\geq\frac{A}{A+22B+22C}+\frac{B}{B+22A+22C}+\frac{C}{C+22A+22B}.$$
Note that
$$\frac{A}{A+22B+22C}=\frac{22}{21}\frac{A+B+C}{A+22B+22C}-\frac{1}{21}$$
and similarly for the other two fractions. Thus
$$\frac{A}{A+22B+22C}+\frac{B}{B+22A+22C}+\frac{C}{C+22A+22B}=\frac{22}{21}\left(\frac{A+B+C}{A+22B+22C}+\frac{A+B+C}{B+22A+22C}+\frac{A+B+C}{C+22A+22B}\right)-\frac{1}{7}.$$
By the inequality (1) we have
$$\frac{A+B+C}{A+22B+22C}+\frac{A+B+C}{B+22A+22C}+\frac{A+B+C}{C+22A+22B}\geq\frac{9}{\frac{A+22B+22C}{A+B+C}+\frac{B+22A+22C}{A+B+C}+\frac{C+22A+22B}{A+B+C}}=\frac{9}{45}=\frac{1}{5}.$$
Putting these inequalities together, we see that
$$\frac{1}{\sqrt{1+2024a}}+\frac{1}{\sqrt{1+2024b}}+\frac{1}{\sqrt{1+2024c}}\geq\frac{22}{21}\cdot\frac{1}{5}-\frac{1}{7}=\frac{1}{15}.$$
The equality holds if and only if $a=b=c=1$. This completes our first solution.

Second Solution. Our first solution, while in principle completely elementary, requires several non obvious manipulations. In the second solution, we will use multivariable calculus to get a more straightforward argument. We will prove the following more general result.

Theorem 1. *Let $t\geq 8$ be a real number. Then*
$$\frac{1}{\sqrt{1+ta}}+\frac{1}{\sqrt{1+tb}}+\frac{1}{\sqrt{1+tc}}\geq\frac{3}{\sqrt{1+t}}$$

for any positive real numbers a, b, c such that $abc = 1$. Equality holds if and only if $a = b = c = 1$.

In order to prove Theorem 1, consider the function $f(x) = \dfrac{1}{\sqrt{1+tx}}$ and define

$$H(x,y) = f(x) + f(y) + f\left(\frac{1}{xy}\right).$$

Theorem 1 is equivalent to the statement that $H(1,1)$ is the smallest value of H on the set of pairs of positive real numbers and $(1,1)$ is the only point at which the minimum is attained.

Suppose that H attains its smallest value at some point (u,w). Then

$$\frac{\partial H}{\partial x}(u,w) = 0 = \frac{\partial H}{\partial y}(u,w). \tag{3}$$

Note that

$$\frac{\partial H}{\partial x}(x,y) = f'(x) - \frac{1}{x^2 y} f'\left(\frac{1}{xy}\right)$$

and

$$\frac{\partial H}{\partial y}(x,y) = f'(y) - \frac{1}{x y^2} f'\left(\frac{1}{xy}\right).$$

Thus (3) is equivalent to the equalities

$$uf'(u) = \frac{1}{uw} f'\left(\frac{1}{uw}\right) = wf'(w).$$

Let $G(x) = xf'(x) = -tx(1+tx)^{-3/2}/2$. Then we have $G(u) = G(w) = G(1/uw)$. Note that $G'(x) = t(1+tx)^{-5/2}(tx-2)/4$, so $G'(x) < 0$ for $x \in (0, 2/t)$ and $G'(x) > 0$ for $x > 2/t$. Thus G is decreasing on $(0, 2/t)$ and increasing on $(2/t, \infty)$. It follows that for any given s the equation $G(x) = s$ has at most two different solutions. Since $G(u) = G(w) = G(1/uw)$, we must have $u = w$ or $u = 1/uw$, or $w = 1/uw$. If $u = w$ then $H(u,u)$ is the smallest value of H. Note that $H(x,y) = H(x, 1/xy) = H(y, 1/xy)$ for any x, y. Thus, if $u = 1/uw$ then $H(u,u)$ is again the smallest value of H. Finally, if $w = 1/uw$ then $H(w,w)$ is the smallest value of H.

Consider now the function $S(x) = H(x,x) = 2f(x) + f(x^{-2})$. We showed that S attains its smallest value at u or w. Now

$$S'(x) = 2f'(x) - 2x^{-3} f'(x^{-2}) = -t(1+tx)^{-3/2} + tx^{-3}(1+tx^{-2})^{-3/2}.$$

Thus $S'(x) = 0$ iff $1 + tx = x^2(1 + tx^{-2}) = x^2 + t$, i.e. $x^2 - tx + t - 1 = 0$. This equation has 2 solutions: $x = 1$ and $x = t - 1$. Moreover, $S'(x) > 0$ if and only if $x \in (1, t-1)$, so S decreases on $(0,1)$, increases on $(1, t-1)$, and decreases again on $(t-1, \infty)$. Since $\lim_{x\to\infty} S(x) = 1$ and $S(1) \le 1$, S attains its smallest value at $x = 1$ and this is the only minimum of S. Thus, either $u = 1$ and $u = w$ or $u = 1/uw$, or $w = 1$ and $w = 1/uw$. In either case we get $u = 1 = w$.

We showed that if H attains its smallest value at some point (u,w) then $u = w = 1$. We still need to show that H actually attains its smallest value at some point. To this end, let M be the infimuum of the set $\{H(x,y) : x > 0, y > 0\}$ of all values of H. Since $t \ge 8$, we have $M \le H(1,1) \le 1$. If $M = 1$, then $H(1,1)$ is the smallest value of H. Suppose that $M < 1$. There is a sequence (x_n, y_n) such that $\lim_{n\to\infty} H(x_n, y_n) = M$. Passing to a subsequence if necessary, we may assume that $\lim_{n\to\infty} x_n = u$ and $\lim_{n\to\infty} y_n = w$, where $u, w \in [0, \infty]$. If $u = 0$ then, since $H(x_n, y_n) \ge f(x_n)$ and $\lim_{n\to\infty} f(x_n) = f(0) = 1$, we get $M \ge 1$, a contradiction. Similarly, neither $w = 0$ nor $\lim_{n\to\infty} 1/(x_n y_n) = 0$ is possible. If we had $u = \infty$, then either $w = 0$ or $\lim_{n\to\infty} 1/(x_n y_n) = 0$, neither of which is possible. Thus u must be finite and positive. Similarly, w is finite and positive. Since H is continuous, we have $M = H(u,w)$, so H attains its smallest value at (u,w). As we have seen, this implies that $u = w = 1$.

We proved that $H(1,1)$ is the smallest value of H and $H(u,w) = H(1,1)$ iff $u = w = 1$. This completes the proof of Theorem 1.

Third Solution (after Andrew Zhou). Andrew Zhou's solution is based on a general result in elementary inequalities. To state it we need to recall the notion of a convex (concave) function. A function $f : I \longrightarrow \mathbb{R}$ is called **convex** if for any $x, y \in I$ and any $t \in [0, 1]$ we have the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

Here I is any inteval (can be open, or closed, or half open; bounded or unbounded). We say that f is strictly convex if the above inequality is strict unless $x = y$ or $t = 0$, or $t = 1$. We say that f is (strictly) **concave** if the function $-f$ is (strictly) convex. A twice-differentiable function is convex on I if and only if its second derivative is non-negative on I (if the derivative is positive, the function is strictly convex). See the solution to problem 3 from Fall 2022 for some additional discussion of convex functions. The following result, known as **Karamata's inequality** (or **majorization inequality**) is a very useful tool for studying inequalities.

Karamata's Inequality. *Let $f : I \longrightarrow \mathbb{R}$ be a convex function on an interval I. Suppose that $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ are numbers in I such that*

- $x_1 \ge x_2 \ge \ldots \ge x_n$ *and* $y_1 \ge y_2 \ge \ldots \ge y_n$.
- $x_1 + \ldots + x_i \ge y_1 + \ldots + y_i$ *for* $i = 1, 2, \dots, n-1$.
- $x_1 + \ldots + x_n = y_1 + \ldots + y_n$.

Then

$$f(x_1) + \ldots + f(x_n) \ge f(y_1) + \ldots + f(y_n). \tag{4}$$

If f is strictly convex, then equality in (4) holds if and only if $x_i = y_i$ for $i = 1, 2, \dots, n$.

It should be clear that for concave functions the inequality in (4) should be reversed.

Note that the choice of $y_1 = y_2 = \ldots = y_n = (x_1 + \ldots + x_n)/n$ satisfies the assumptions of Karamata's inequality. Thus the following inequality, known as Jensen's inequality, is a special case of Karamata's inequality:

Jensen's Inequality. *Let $f : I \longrightarrow \mathbb{R}$ be a convex function. Suppose that $x_1, x_2, \dots, x_n$ are in I. Then*

$$\frac{f(x_1) + \ldots + f(x_n)}{n} \ge f\left(\frac{x_1 + \ldots + x_n}{n}\right).$$

Note that applying Jensen's inequality to the convex function $-\ln x$ yields the (logarithm of) AM-GM ineuqlity. For more about the Karamata's ineuqality and some related topics we recommend the following paper:

INEQUALITIES OF KARAMATA, SCHUR AND MUIRHEAD, AND SOME APPLICATIONS by Zoran Kadelburg, Dusan Dukić, Milivoje Lukić and Ivan Matić.

Here is a clickable link:

Inequalities of Karamata ...

Suppose now that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function with exactly one inflection point at $x = u$. This means that f is concave on $(-\infty, u]$ and convex on $[u, \infty)$ (or the other way, convex on $(-\infty, u]$ and concave on $[u, \infty)$). Consider any real numbers $y_1 \ge y_2 \ge \ldots \ge y_n$. Let k be such that $y_k \ge u \ge y_{k+1}$. Let $x_1 = y_1 + \ldots + y_k - (k-1)u$, $x_2 = \ldots = x_k = u$. It is easy to see that $x_1, \dots, x_k$ and $y_1, \dots, y_k$ satisfy the assumptions of Karamata's inequality for f which is convex on $I = [u, \infty)$. Thus $f(x_1) + \ldots + f(x_k) \ge f(y_1) + \ldots + f(y_k)$, i.e.

$$f(y_1 + \ldots + y_k - (k-1)u) + (k-1)f(u) \ge f(y_1) + \ldots + f(y_k).$$

Since f is concave on $(-\infty, u]$, we see from Jensen's inequality that

$$(k-1)f(u) + f(y_{k+1}) + \ldots + f(y_n) \le (n-1)f\left(\frac{(k-1)u + y_{k+1} + \ldots + y_n}{n-1}\right).$$

Combining the last two inequalities, we get

$$f(y_1) + \ldots + f(y_n) \le f(y_1 + \ldots + y_k - (k-1)u) + (n-1)f\left(\frac{(k-1)u + y_{k+1} + \ldots + y_n}{n-1}\right).$$

Note that we assumed that there is k such that $y_k \ge u \ge y_{k+1}$. If no such k exists then either $y_n > u$ or $y_1 < u$. In the former case, we have

$$f(y_1) + \ldots + f(y_n) \le f(y_1 + \ldots + y_n - (n-1)u) + (n-1)f(u).$$

In the latter case, we have

$$f(y_1) + \ldots + f(y_n) \le nf\left(\frac{y + \ldots + y_n}{n}\right) = f\left(\frac{y + \ldots + y_n}{n}\right) + (n-1)f\left(\frac{y + \ldots + y_n}{n}\right).$$

The conclusion of our discussion so far is that for any $y_1, \ldots, y_n$ there exist p and q such that $y_1 + \ldots + y_n = p + (n-1)q$ and $f(y_1) + \ldots + f(y_n) \le f(p) + (n-1)f(q)$. We showed it under the assumption that $y_1 \ge \ldots \ge y_k$, but we can always do that since the sum $f(y_1) + \ldots + f(y_n)$ does not depend on the order.

Applying this argument to the function $-f(-x)$ (which is concave on $(-\infty, -u]$ and convex on $[-u, \infty)$) and the numbers $-y_1, \ldots, -y_n$ we see that there are p_1, q_1 such that $y_1 + \ldots + y_n = p_1 + (n-1)q_1$ and $f(y_1) + \ldots + f(y_n) \ge f(p_1) + (n-1)f(q_1)$.

An immediate corollary of the above observations is the follwing result, called the $N-1$ equal value principle:

N-1 Equal Value Principle. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function with exactly one inflection point and let c be a real number. Suppose that the function $h(t) = (n-1)f(t) + f(c - (n-1)t)$ asssumes its largest (smallest) value M at some point $t \in \mathbb{R}$. Then M is the largest (smallest) value of the function $H(x_1, \ldots, x_n) = f(x_1) + \ldots + f(x_n)$ considered on the set of all $(x_1, \ldots, x_n)$ such that $x_1 + \ldots + x_n = c$.*

In other words, to show that $f(x_1) + \ldots + f(x_n) \ge M$ for all $x_1, \ldots, x_n$ such that $x_1 + \ldots + x_n = c$, it suffices to show that $h(t) \ge M$ for all t.

For various variations of the equal value principle we recommend the paper

THE EQUAL VARIABLE METHOD by Vasile Cirtoaje
Here is a clickable link: The Equal Variable Method

In order to apply the N-1 Equal Value Principle to our problem, we take $n = 3$, $f(x) = 1/\sqrt{1 + 2024e^x}$, $M = 1/15$, and $c = 0$. Then f has one inflection point. The fact that $h(t) \ge 1/15$ is proved as in our second solution. Setting $a = e^x$, $b = e^y$, $c = e^z$ gives the conclusion of our problem.

Solution 4 (after Josiah Moltz). Josiah's solution starts with the following observation.

Proposition. *Let $t \ge 3$ be a real number. Then*

$$\frac{1}{\sqrt{1 + ta_1}} + \frac{1}{\sqrt{1 + ta_2}} \ge \frac{2}{\sqrt{1 + t}}$$

for any positive real numbers a_1, a_2 such that $a_1 a_2 = 1$. Equality holds if and only if $a_1 = a_2 = 1$.

This can be justified by using starightforward calculus techniques to show that for $t \ge 3$ the function

$$f_t(x) = \frac{1}{\sqrt{1 + tx}} + \frac{1}{\sqrt{1 + \frac{t}{x}}}$$

assumes its smallest value when $x = 1$. Alternatively, follow the ideas of Problem 3 below to show (6) for $n = 1$ and then derive Theorem 2 for $n = 2$. The proposition is easily seen equivalent to the inequality

$$\frac{1}{\sqrt{1 + x}} + \frac{1}{\sqrt{1 + y}} \ge \frac{2}{\sqrt{1 + \sqrt{xy}}}$$

for any positive x, y such that $xy \geq 9$ (just take $t = \sqrt{xy}$, $a_1 = x/t$, $a_2 = y/t$).

Suppose now that $abc = 1$ and $a \geq b \geq c$. Then $ab \geq 1$ and $(2024a)(2024b) \geq 2024^2 \geq 9$. Thus we have

$$\frac{1}{\sqrt{1+2024a}} + \frac{1}{\sqrt{1+2024b}} \geq \frac{2}{\sqrt{1+2024\sqrt{ab}}}.$$

Thus it would suffice to show that

$$\frac{2}{\sqrt{1+2024\sqrt{ab}}} + \frac{1}{\sqrt{1+2024c}} \geq \frac{1}{15}$$

which can be done by showing that the function

$$f(x) = \frac{2}{\sqrt{1+2024x}} + \frac{1}{\sqrt{1+\frac{2024}{x^2}}}$$

assumes its smallest value for $x = 1$ by using standard calculus (as we did in our second and third solutions). Josiah proceeds differently though. Suppose first that $(2024c)(2024) = 2024^2 c \geq 9$. Then

$$\frac{1}{\sqrt{1+2024c}} + \frac{1}{\sqrt{1+2024}} \geq \frac{2}{\sqrt{1+2024\sqrt{c}}}.$$

Note that $(2024\sqrt{ab})(2024\sqrt{c}) = 2024^2 \geq 9$ so

$$\frac{1}{\sqrt{1+2024\sqrt{ab}}} + \frac{1}{\sqrt{1+2024\sqrt{c}}} \geq \frac{2}{\sqrt{1+2024\sqrt{\sqrt{ab}\sqrt{c}}}} = \frac{2}{\sqrt{2025}} = 2/45.$$

Putting all these inequalities together we get

$$\frac{1}{\sqrt{1+2024a}} + \frac{1}{\sqrt{1+2024b}} + \frac{1}{\sqrt{1+2024c}} + \frac{1}{\sqrt{1+2024}} \geq \frac{2}{\sqrt{1+2024\sqrt{ab}}} + \frac{2}{\sqrt{1+2024\sqrt{c}}} \geq \frac{4}{45}$$

from which we get

$$\frac{1}{\sqrt{1+2024a}} + \frac{1}{\sqrt{1+2024b}} + \frac{1}{\sqrt{1+2024c}} \geq \frac{3}{45} = \frac{1}{15}.$$

It remians to consider the case when $c < 9/2024^2$. But then

$$\frac{1}{\sqrt{1+2024c}} \geq \frac{1}{\sqrt{1+\frac{9}{2024}}} > \frac{1}{15}$$

so clearly

$$\frac{1}{\sqrt{1+2024a}} + \frac{1}{\sqrt{1+2024b}} + \frac{1}{\sqrt{1+2024c}} \geq \frac{1}{15}.$$

This completes Josiah's solution.

We end our discussion with several problems which expand the methods of our first solution. We start with the following more general version of the AM-GM inequality.

Extended AM-GM Inequality. *If $a_1, \ldots, a_n$ are positive real numbers then*

$$s_1 a_1 + \ldots + s_n a_n \geq a_1^{s_1} a_2^{s_2} \ldots a_n^{s_n}$$

for any non-negative numbers $s_1, \ldots, s_n$ such that $s_1 + s_2 + \ldots + s_n = 1$. Equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Problem 1. Prove the Extended AM-GM inequality. Hint: First use the AM-GM inequality to prove a special case of the Extended AM-GM inequality when $n = 2$ and $s_1 = k/m$ is a rational number. From this conclude that the Extended AM-GM inequality holds for $n = 2$ and any $s_1 \in [0, 1]$ ($s_2 = 1 - s_1$). Then use induction on n.

Problem 2. Prove that for any positive real numbers u, w and any $\lambda > 1$ we have

$$\sqrt{1+(\lambda^2-1)uw} \leq 1 + \frac{\lambda-1}{2}u^{\frac{\lambda+1}{\lambda}} + \frac{\lambda-1}{2}w^{\frac{\lambda+1}{\lambda}} \tag{5}$$

and the equality holds if and only if $u = w = 1$. Use this to extend our first solution to a proof of Theorem 1.

Problem 3. Prove that for any positive real numbers $u_1, u_2, \ldots, u_n$ and any $t > 0$ we have

$$\sqrt{1+tu_1u_2\ldots u_n} \leq 1 + su_1^\lambda + su_2^\lambda + \ldots + su_n^\lambda \tag{6}$$

where $s = (\sqrt{t+1}-1)/n$ and $\lambda = n(2+sn)/(2+2sn)$. Equality holds if and only if $u_1 = \ldots = u_n = 1$. Use this to prove the following extension of Theorem 1:

Theorem 2. *Let $t \geq n^2 - 1$ be a real number. Then*

$$\frac{1}{\sqrt{1+ta_1}} + \frac{1}{\sqrt{1+ta_2}} + \ldots + \frac{1}{\sqrt{1+ta_n}} \geq \frac{n}{\sqrt{1+t}}$$

for any positive real numbers $a_1, a_2, \ldots, a_n$ such that $a_1a_2\ldots a_n = 1$. Equality holds if and only if $a_1 = a_2 = \ldots = a_n = 1$.

Problem 4. Push the methods used above further and show that that for any positive real numbers a, b, c such that $abc = 1$ and any $\lambda \geq 26$ we have

$$\frac{1}{\sqrt[3]{1+\lambda a}} + \frac{1}{\sqrt[3]{1+\lambda b}} + \frac{1}{\sqrt[3]{1+\lambda c}} \geq \frac{3}{\sqrt[3]{1+\lambda}}. \tag{7}$$

Try to generalize further.