Problem 1. For a positive integer n, let $s(n)$ be the square of the sum of the decimal digits of n. For example, $s(134) = (1+3+4)^2 = 64$. Find all n such that $n = s^k(n)$ for some k, where $s^k = s \circ s \circ \ldots \circ s$ is the composition of s with itself k-times.

Solution. We start with the following simple observation.

Lemma. For any integer $m \geq 5$ we have $m^2 < 10^{m-3}$.

We prove the lemma by induction on m. For $m = 5$ the lemma is clear: $25 < 10^2$. Assuming that $m^2 < 10^{m-3}$ we see that

$$(m+1)^2 = m^2 + 2m + 1 < 10m^2 < 10 \cdot 10^{m-3} = 10^{(m+1)-3}.$$

Hence the lemma is true by the method of mathematical induction.

We claim now that $s(n) < n$ for every $n \geq 10^4$. Indeed if $n \geq 10^4$, then $10^{m-1} \leq n < 10^m$ for some $m \geq 5$. This means that n is an m-digit number and therefore using the lemma we have

$$s(n) \leq (9m)^2 = 81m^2 < 10^2 m^2 < 10^2 \cdot 10^{m-3} = 10^{m-1} \leq n.$$

We claim now that $s(n) < n$ for all $n \geq 36^2 = 1296$. In fact, if $n \geq 10^4$ then we already know this. If $1296 < n < 10^4$ then n is a four-digit number so

$$s(n) \leq (4 \cdot 9)^2 = 36^2 = 1296 < n.$$

We claim now that $s(n) < n$ for all $n > 27^2 = 729$. Indeed, we already proved this if $n > 1296$. For $729 < n \leq 1296$ we have $s(n) \leq (9+9+9)^2 = 27^2 < n$.

We claim now that $s(n) < n$ for all $n > 24^2 = 576$. Indeed, we already proved this if $n > 729$. For $576 < n \leq 729$ we have $s(n) \leq (6+9+9)^2 = 24^2 < n$.

We claim now that $s(n) < n$ for all $n > 22^2 = 484$. Indeed, we already proved this if $n > 576$. For $484 < n \leq 576$ we have $s(n) \leq (4+9+9)^2 = 22^2 < n$.

We claim now that $s(n) < n$ for all $n > 20^2 = 400$. Indeed, we already proved this if $n > 484$. For $400 < n \leq 484$ we have $s(n) \leq (4+7+9)^2 = 20^2 < n$.

Note that $s(399) = 21^2 > 399$ and $s(400) = 16 < 400$. Thus $s(n) < n$ for all $n \geq 400$.

Suppose now that $n = s^k(n)$ for some $k \geq 1$. Then one of the numbers $n, s(n), \ldots, s^k(n)$ must be smaller than 400. Indeed, otherwise we would have $n > s(n) > \ldots > s^k(n) = n$, which is not possible. Now for each $n = 1^2, 2^2, \ldots, 19^2$ we compute the sequence $n, s(n), s^2(n), s^3(n), \ldots$ and observe that in each case we end up with $1, 1, ...$, or $81, 81, ...$, or $169, 256, 169, 256, ...$ Thus only 4 numbers n satisfying the conditions of the problem: $1, 81$ when $k = 1$ and $169, 256$ when $k = 2$.

Problem. It is not hard to see that for any n the sequence $n, s(n), s^2(n), s^3(n), \ldots$ will eventually hit 1, or 81, or 169. Let $k(n)$ be the smallest integer such that $s^{k(n)}(n) \in \{1, 81, 169, 256\}$ and let $t(n) = s^{k(n)}(n)$. What can be said about the function $k(n)$? Suppose that we choose n randomly. Which value of $t(n)$ is most likely?