Problem 5. Prove that for $\alpha \in (0, \pi/2)$ we have

$$\tan(\alpha) - \tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\alpha}{4}\right) - \tan\left(\frac{\alpha}{8}\right) + \ldots \geq \tan\left(\frac{2\alpha}{3}\right).$$

Solution. First note that by the alternating series test the left hand side is indeed a convergent series (it is actually absolutely convergent). We are asked to show that

$$\tan(\alpha) - \tan\left(\frac{2\alpha}{3}\right) \geq \tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\alpha}{4}\right) + \tan\left(\frac{\alpha}{8}\right) - \ldots = \sum_{k=1}^{\infty}\left(\tan\left(\frac{\alpha}{2^{2k-1}}\right) - \tan\left(\frac{\alpha}{2^{2k}}\right)\right).$$

By the mean value theorem, we have

$$\tan(\alpha) - \tan\left(\frac{2\alpha}{3}\right) = \left(\alpha - \frac{2\alpha}{3}\right)\frac{1}{\cos^2\phi} = \frac{\alpha}{3\cos^2\phi}$$

for some $\phi \in (\alpha, 2\alpha/3)$. By the same theorem, we have

$$\tan\left(\frac{\alpha}{2^{2k-1}}\right) - \tan\left(\frac{\alpha}{2^{2k}}\right) = \left(\frac{\alpha}{2^{2k-1}} - \frac{\alpha}{2^{2k}}\right)\frac{1}{\cos^2\phi_k} = \frac{\alpha}{2^{2k}\cos^2\phi_k}$$

for some $\phi_k \in (\alpha/2^{2k-1}, \alpha/2^{2k})$. Note that $\pi/2 > \phi > \phi_1 > \phi_2 > \ldots > 0$. Thus $1/\cos^2(\phi) > 1/\cos^2(\phi_1) > 1/\cos^2(\phi_2) > \ldots$ and therefore

$$\sum_{k=1}^{\infty}\left(\tan\left(\frac{\alpha}{2^{2k-1}}\right) - \tan\left(\frac{\alpha}{2^{2k}}\right)\right) = \sum_{k=1}^{\infty}\frac{\alpha}{2^{2k}\cos^2\phi_k} < \sum_{k=1}^{\infty}\frac{\alpha}{2^{2k}\cos^2\phi} == \frac{\alpha}{3\cos^2\phi} = \tan(\alpha) - \tan\left(\frac{2\alpha}{3}\right)$$

(we used the equality $\sum_{k=1}^{\infty} 1/2^{2k} = 1/3$).

Second solution (after Gerald Marchesi). Let $c_0 = 1$ and

$$c_k = 1 - \frac{1}{4} - \frac{1}{4^2} - \ldots - \frac{1}{4^k} = \frac{2 + 3^{-k}}{3}$$

for $k > 0$. Clearly (c_k) is a decreasing sequence converging to $2/3$. In particular, $c_{k+1} = c_k - 1/4^{k+1}$ and $c_k > 2/3 > 1/2^{2k+1}$ for every k.

Note that for any $\phi \in (0, \pi/2)$ the function $f(x) = \tan x - \tan(x - \phi)$ is increasing on $(\phi, \pi/2)$. This follows from the fact that $f'(x) = \sec^2(x) - \sec^2(x - \phi)$ is positive on $(\phi, \pi/2)$. Taking $\phi = \alpha/4^{k+1}$, we get $f(c_k\alpha) > f(\alpha/2^{2k+1})$, i.e.

$$\tan(c_k\alpha) - \tan(c_{k+1}\alpha) > \tan\left(\frac{\alpha}{2^{2k+1}}\right) - \tan\left(\frac{\alpha}{2^{2k+2}}\right).$$

Note now that for any positive integer n we have

$$\tan\alpha = \tan(c_{n+1}\alpha) + \sum_{k=0}^{n}(\tan(c_k\alpha) - \tan(c_{k+1}\alpha)) > \tan\left(\frac{2\alpha}{3}\right) + \sum_{k=0}^{n}\left(\tan\left(\frac{\alpha}{2^{2k+1}}\right) - \tan\left(\frac{\alpha}{2^{2k+2}}\right)\right).$$

Letting n go to infinity, we get

$$\tan(\alpha) - \tan\left(\frac{2\alpha}{3}\right) > \sum_{k=0}^{\infty}\left(\tan\left(\frac{\alpha}{2^{2k+1}}\right) - \tan\left(\frac{\alpha}{2^{2k+2}}\right)\right)$$

as required.

Third solution (after Matt Wolak). Consider the function $f(x) = \tan(x) - x\sec^2(2\alpha/3)$. It is a simple exercise to show that f has the following properties:

1. f is odd.

2. f is decreasing on $[-2\alpha/3, 2\alpha/3]$ and increasing on $[2\alpha/3, \pi/2)$.

3. $f(x) > f(0) = 0$ for $x \in [-2\alpha/3, 0)$.

4. $f(x) \geq f(2\alpha/3)$ for $x \in [-2\alpha/3, \pi/2)$.

For any positive integer n we have

$$\sum_{k=0}^{2n+1} f\left(\alpha\left(-\frac{1}{2}\right)^k\right) = f(\alpha) + \sum_{k=0}^{n}\left(f\left(\frac{-\alpha}{2^{2k+1}}\right) - f\left(\frac{-\alpha}{2^{2k+2}}\right)\right).$$

We have $f(\alpha) \geq f(2\alpha/3)$ by property 4 and each term in the sum on the right is positive by property 2. It follows that

$$\sum_{k=0}^{2n+1} f\left(\alpha\left(-\frac{1}{2}\right)^k\right) > f\left(\frac{2\alpha}{3}\right)$$

for every positive integer n. On the other hand,

$$\sum_{k=0}^{2n+1} f\left(\alpha\left(-\frac{1}{2}\right)^k\right) = \sum_{k=0}^{2n+1}(-1)^k \tan\left(\frac{\alpha}{2^k}\right) - \sec^2\left(\frac{2\alpha}{3}\right)\sum_{k=0}^{2n+1}(-1)^k\frac{\alpha}{2^k}.$$

Letting n go to infinity and using the fact that $\sum_{k=0}^{\infty}(-1)^k\alpha/2^k = 2\alpha/3$, we get

$$\sum_{k=0}^{\infty}(-1)^k \tan\left(\frac{\alpha}{2^k}\right) - \sec^2\left(\frac{2\alpha}{3}\right)\frac{2\alpha}{3} > f\left(\frac{2\alpha}{3}\right) = \tan\left(\frac{2\alpha}{3}\right) - \sec^2\left(\frac{2\alpha}{3}\right)\frac{2\alpha}{3}$$

i.e.

$$\sum_{k=0}^{\infty}(-1)^k \tan\left(\frac{\alpha}{2^k}\right) > \tan\left(\frac{2\alpha}{3}\right).$$

Fourth solution. We will use Karamata's inequality (see page 4 of the solution to problem 6 from Spring 2025). Let $n \geq 2$ be a positive integer. Let $x_k = \alpha/2^{2k-2}$ for $k = 1, \ldots, n$. Let $y_k = \alpha/2^{2k-3}$ for $k = 2, \ldots n$ and let $y_1 = (x_1 + \ldots + x_n) - (y_2 + \ldots + y_n)$. Then $x_1 > \ldots > x_n$, $y_1 > \ldots > y_n$, $x_1 + \ldots + x_n = y_1 + \ldots + y_n$, and $x_1 + \ldots + x_k \geq y_1 + \ldots + y_k$ for $k = 1, \ldots, n$. Since $\tan x$ is convex on $(0, \pi/2)$, Karamata's inequality says that

$$\tan x_1 + \ldots + \tan x_n \geq \tan y_1 + \ldots + \tan y_n.$$

In other words,

$$\sum_{k=0}^{2n-2}(-1)^k \tan\left(\frac{\alpha}{2^k}\right) \geq \tan y_1 = \tan\left(\sum_{k=0}^{2n-2}(-1)^k\frac{\alpha}{2^k}\right).$$

Letting n go to infinity, we get

$$\sum_{k=0}^{\infty}(-1)^k \tan\left(\frac{\alpha}{2^k}\right) \geq \tan\left(\frac{2\alpha}{3}\right).$$

Problem. Let f be a convex function on $[0, a]$ and let n be a positive integer. Let $a \geq a_1 \geq a_2 \geq \ldots \geq a_{2n-1} \geq 0$.

a) Use Karamata's inequality to prove that

$$\sum_{k=1}^{2n-1}(-1)^{k-1} f(a_k) \geq f\left(\sum_{k=1}^{2n-1}(-1)^{k-1}a_k\right)$$

b) Suppose that, in addition, $f(0) \leq 0$. Prove that

$$\sum_{k=1}^{2n-2}(-1)^{k-1} f(a_k) \geq f\left(\sum_{k=1}^{2n-2}(-1)^{k-1}a_k\right)$$