Problem 3. 24 friends regularly attend parties organized by a puzzle-loving host. The host challenges the group to the following game. The host has 24 name tags with the names of the 24 friends and she randomly distributes one name tag to each of the friends. The goal of the game is for each of the friends to find out who has their name tag. Each of the 24 players is allowed to ask the host in private (so no one else can hear it) about the name tags of up to 11 friends. No other communication about the name tags is allowed. If each of the 24 friends correctly identifies the person holding their name tag, the host provides a ride home for each of them. Otherwise they all need to arrange for their ride home. Design as good a strategy as you can for the friends to maximize the likelihood to receive a free ride home.

Solution. For simplicity, we number the 24 friends $1, 2, \dots, 24$ and use their number instead of their name. The host picks randomly a permutation f of the set $\{1, 2, \dots, 24\}$ and gives person i the number $f(i)$. Here is the strategy we propose. Person i asks the host about the number given to $f(i)$ (note that i knows $f(i)$). The host answers giving i the number $f(f(i)) = f^2(i)$. If $f^2(i) = i$ then i knows that $f(i)$ has name tag i. Otherwise i asks the host about the number given to $f^2(i)$ and gets the answer $f(f^2(i)) = f^3(i)$. Again, if $f^3(i) = i$ then i knows that $f^2(i)$ has name tag i. Otherwise i asks the host about the numbers given to $f^3(i)$, and so on. If i never finds out in this way who has their name tag, this means that none of the numbers $f(i), f^2(i), \dots, f^{12}(i)$ is i. In this case i makes a guess that $f^{12}(i)$ has their name tag, i.e. that $f^{13}(i) = i$.

Using the above strategy, the group will win the game if and only if for every i one of the numbers $f(i), f^2(i), \dots, f^{13}(i)$ is equal to i. We will now compute the probability that this happens. To this end, we need to recall some basic observations about permutations.

Let g be a permutation of a finite set S, i.e. g is a bijective function $g : S \longrightarrow S$.

- for any $s \in S$ the sequence $g(s), g^2(s), g^3(s), \dots$ contains s.
- if k is smallest such that $g^k(s) = s$ then the elements $g(s), g^2(s), \dots, g^k(s)$ are all distinct. The set $\{g(s), g^2(s), \dots, g^k(s)\}$ is called **the orbit** of s under g.
- if s, t are two elements of S then their orbits are either equal or they are disjoint. Thus the set S is partitioned into disjoint orbits, called orbits of g.

Returning to our problem, we see that the strategy works for the group if and only if all orbits of the permutation f have at most 13 elements. We will now count permutations of $\{1, 2, \dots, 24\}$ which do not have this property. i.e. permutations which have an orbit of length at least 14. Note that any permutation of $\{1, 2, \dots, 24\}$ can have at most one orbit longer that 12 (since orbits are disjoint). In order to enumerate the permutations with an orbit longer than 13, we need to choose the length $k \in \{14, 15, \dots, 24\}$ of such an orbit, then choose which subset T of $\{1, 2, \dots, 24\}$ forms this orbit, then specify how our permutation acts on T and how it permutes the complement of T. We have $\binom{24}{k}$ possibilities to choose T. Then there are $(k-1)!$ ways to turn T into an orbit (by taking the smallest element $t \in T$ and then specifying the sequence $f(t), f^2(t), \dots, f^{k-1}(t)$, which is just a permutation of the set $T - \{t\}$). Also, there are $(24-k)!$ ways to specify how our permutation permutes the complement of T. It follows that the number of permutations of the set $\{1, 2, \dots, 24\}$ which have an orbit of length $k > 13$ is equal to

$$\binom{24}{k}(k-1)!(24-k)! = \frac{24!}{k}.$$

Thus, the number of permutations whose all orbits have length at most 13 is

$$24! - \sum_{k=14}^{24} \frac{24!}{k} = 24! \left(1 - \sum_{k=14}^{24} \frac{1}{k}\right)$$

and the probability that our strategy works is

$$1 - \sum_{k=14}^{24} \frac{1}{k} = 0.4041\ldots > 2/5.$$

Exercise. Prove the properties of permutations used in the above solution.

We do not know the answer to the following problem

Problem. Is there a better strategy?