Problem 2. Find all functions $F:\mathbb{R}^2 \longrightarrow \mathbb{R}$ such that

(i) If $ABCD$ is any rectangle on the plane $\mathbb{R}^2$ then $F(A)+F(C)=F(B)+F(D)$;

(ii) The second order partial derivatives $\dfrac{\partial^2 F}{\partial x\partial x}$, $\dfrac{\partial^2 F}{\partial x\partial y}$, $\dfrac{\partial^2 F}{\partial y\partial x}$, $\dfrac{\partial^2 F}{\partial y\partial y}$ exist and are continuous on $\mathbb{R}^2$ (this means that F is of class C^2);

(iii) $F(0,0)=0$, $F(1,0)=1=F(0,1)$, $\dfrac{\partial F}{\partial x}(0,0)=0$.

Solution. Suppose that a function $F:\mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies the following condition

($*$) $F(A)+F(C)=F(B)+F(D)$ for any rectangle $ABCD$ on the plane $\mathbb{R}^2$ whose sides are parallel to the $x-$ and $y-$ axes.

This means that for any real numbers x,y,s,t we have

$$F(x,y)+F(x+s,y+t)=F(x,y+t)+F(x+s,y).$$

In particular, $F(s,t)+F(0,0)=F(s,0)+F(0,t)$ for any real numbers s,t. Let $g(x)=F(x,0)-F(0,0)/2$, $h(x)=F(0,x)-F(0,0)/2$. Then

$$F(x,y)=g(x)+h(y) \text{ and } g(0)=h(0). \tag{1}$$

Conversely, any function F of the form (1) satisfies the condition ($*$).

Suppose now that in addition to condition (1) the function F satisfies $F(0,0)=0$. Then $g(0)+h(0)=0$ and $g(0)=h(0)$, i.e. $g(0)=h(0)=0$.

Let R_α denote the counter-clock rotation of the plane about the origin $(0,0)$ by angle α. In other words,

$$R_\alpha(x,y)=(x\cos\alpha-y\sin\alpha, x\sin\alpha+y\cos\alpha).$$

The function F satisfies condition (i) of the problem if and only if for every α the function $F_\alpha=F\circ R_\alpha$ has property ($*$). Note that

$$F_\alpha(x,y)=g(x\cos\alpha-y\sin\alpha)+h(x\sin\alpha+y\cos\alpha).$$

As we noticed earlier, F_α has property ($*$) if and only if $F_\alpha(x,y)=g_\alpha(x)+h_\alpha(y)$, where $g_\alpha(x)=F_\alpha(x,0)-F_\alpha(0,0)/2=g(x\cos\alpha)+h(x\sin\alpha)$ and $h_\alpha(x)=F_\alpha(0,x)-F_\alpha(0,0)/2=g(-x\sin\alpha)+h(x\cos\alpha)$. Thus F_α has property ($*$) if and only if

$$g(x\cos\alpha-y\sin\alpha)+h(x\sin\alpha+y\cos\alpha)=g(x\cos\alpha)+h(x\sin\alpha)+g(-y\sin\alpha)+h(y\cos\alpha) \tag{2}$$

for every real numbers x,y.

Suppose now that the partial derivatives $\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$ exist on $\mathbb{R}^2$. This is equivalent to the assumption that the functions g and h are differentiable.

There are now several ways to complete the solution.

First method (after Matt Wolak). Differentiating (2) with respect to α we get

$$g'(x\cos\alpha-y\sin\alpha)(-x\sin\alpha-y\cos\alpha)+h'(x\sin\alpha+y\cos\alpha)(x\cos\alpha-y\sin\alpha)=$$

$$-g'(x\cos\alpha)x\sin\alpha+h'(x\sin\alpha)x\cos\alpha-g'(-y\sin\alpha)y\cos\alpha+h'(y\cos\alpha)y\sin\alpha$$

In particular, setting $\alpha=0$ we get

$$-yg'(x)+xh'(y)=xh'(0)-yg'(0).$$

Taking $x=1$ we have

$$h'(y)=(g'(1)-g'(0))y+h'(0)$$

and taking $y = 1$ yields

$$g'(x) = (h'(1) - h'(0))x + g'(0).$$

Taking $x = 1$ in the last equality we see that $g'(1) - g'(0) = h'(1) - h'(0) = a$. Thus derivatives of both g and h are linear functions, so g and h must be quadratic polynomials:

$$g(x) = ax^2 + px + q \text{ and } h(x) = ax^2 + sx + t$$

for some numbers p, q, s, t. Since $g(0) = 0 = h(0)$, we have $q = t = 0$. So far we have only used the equality $F(0,0) = 0$ from (iii). Since $F(1,0) = g(1) = 1$ and $F(0,1) = h(1) = 1$ we get $a+p = 1 = a+s$. Finally, since $\frac{\partial F}{\partial x}(0,0) = g'(0) = 0$, we get $p = 0$. Putting these together, we have $a = 1$, $p = 0 = s$ and $F(x,y) = x^2 + y^2$. It is easy to check that $x^2 + y^2$ indeed satisfies all the conditions of the problem. Indeed, conditions (ii) and (iii) are clear and for (i) it suffices to check that (2) holds when $g(x) = x^2 = h(x)$, which is a simple exercise.

Remark. Note that in this solution we only need existence of the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$.

Second method. In this solution we only need the equality (2) for $\alpha = \pi/4$:

$$g\left(\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right) + h\left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right) = g\left(\frac{\sqrt{2}}{2}x\right) + h\left(\frac{\sqrt{2}}{2}x\right) + g\left(\frac{-\sqrt{2}}{2}y\right) + h\left(\frac{\sqrt{2}}{2}y\right).$$

Since it holds for all x, y, it is equivalent to

$$g(x-y) + h(x+y) = g(x) + h(x) + g(-y) + h(y).$$

Taking derivative with respect to x we get

$$g'(x-y) + h'(x+y) = g'(x) + h'(x).$$

Now the assumption (ii) (just the existence of the second order partial derivatives) implies that g and h are twice differentiable, so we can take derivative with respect to y of the last equation to get

$$-g''(x-y) + h''(x+y) = 0.$$

Setting $x = y$ we see that h'' is a constant function, and therefore g'' is also constant: $g''(x) = a = h''(x)$. It follows that g and h are quadratic polynomials with the same leading coefficient. Now we proceed as in the last steps of the first method.

Remark. Note that in this solution we need existence of the second order partial derivatives, but we do not need their continuity and we only use the assumption that F_0 and $F_{\pi/4}$ have property $(*)$.

Third method. This method only needs (iii), the existence of the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$, and the assumption that F_0 and F_α have property $(*)$ for a single $\alpha \in (0, \pi/2)$.

Differentiating (2) with respect to y we get

$$-g'(x\cos\alpha - y\sin\alpha)\sin\alpha + h'(x\sin\alpha + y\cos\alpha)\cos\alpha = -g'(y\sin\alpha)\sin\alpha + h'(y\cos\alpha)\cos\alpha. \tag{3}$$

Taking $y = 0$ in (3) we get

$$-g'(x\cos\alpha)\sin\alpha + h'(x\sin\alpha)\cos\alpha = -g'(0)\sin\alpha + h'(0)\cos\alpha = A$$

Setting $y = x\sin\alpha$ we can rewrite the last equation as

$$h'(y) = \tan\alpha g'(y\cot\alpha) + B, \tag{4}$$

where $B = A/\cos\alpha$ is a constant. This allows us to write (3) as follows:

$$-g'(x\cos\alpha - y\sin\alpha)\sin\alpha + (\tan\alpha g'((x\sin\alpha + y\cos\alpha)\cot\alpha) + B)\cos\alpha =$$

$$-g'(y\sin\alpha)\sin\alpha + (\tan\alpha g'(y\cos\alpha\cot\alpha) + B)\cos\alpha.$$

i.e.

$$-g'(x\cos\alpha - y\sin\alpha) + g'(x\cos\alpha + y\cos\alpha\cot\alpha) = -g'(y\sin\alpha) + g'(y\cos\alpha\cot\alpha). \tag{5}$$

Since (5) holds for any x, y, we can plug $x + z/\cos\alpha$ for x to get

$$-g'(x\cos\alpha - y\sin\alpha + z) + g'(x\cos\alpha + y\cos\alpha\cot\alpha + z) = -g'(y\sin\alpha) + g'(y\cos\alpha\cot\alpha). \tag{6}$$

From (5) and (6) we get

$$-g'(x\cos\alpha - y\sin\alpha + z) + g'(x\cos\alpha + y\cos\alpha\cot\alpha + z) = -g'(x\cos\alpha - y\sin\alpha) + g'(x\cos\alpha + y\cos\alpha\cot\alpha)$$

i.e.

$$g'(x\cos\alpha - y\sin\alpha + z) - g'(x\cos\alpha - y\sin\alpha) = g'(x\cos\alpha + y\cos\alpha\cot\alpha + z) - g'(x\cos\alpha + y\cos\alpha\cot\alpha). \tag{7}$$

Note now that given any real numbers u, w we can find x, y such that

$$u = x\cos\alpha - y\sin\alpha \text{ and } w = x\cos\alpha + y\cos\alpha\cot\alpha$$

so

$$g'(u + z) - g'(u) = g'(w + z) - g'(w)$$

holds for all real numbers u, w, z. Taking $w = 0$ and using the fact that $g'(0) = 0$ we see that $g'(u+z) = g'(u) + g'(z)$ for any real numbers u, z. In other words g' is an additive function. We will use now the result that any integrable additive function is linear, i.e. $g'(x) = ax$ for some constant a. Note that (4) implies that $h'(y) = ay + B$ is also linear. Now we complete the solution as in method 1.

Remark. For more information about additive functions see the remark at the end of the solution to Problem 3 from the Fall 2022 problem of the week.

4th method. In this solution we only assume that F is continuous, $F(0,0) = 0$, $F(1,0) = F(-1,0) = 1$, $F(0,1) = F(0,-1)$, and both F_0 and $F_{\pi/4}$ have property $(*)$. In other words, $F(x,y) = g(x) + h(y)$, where g, h are continuous functions such that $g(0) = h(0) = 0$, $g(1) = g(-1) = 1$, $h(1) = h(-1)$ and

$$g(x - y) + h(x + y) = g(x) + h(x) + g(-y) + h(y) \tag{8}$$

(see the second method where we showed that the last equality is equivalent to property $(*)$ for $F_{\pi/4}$). Switching x and y in (8) we get

$$g(y - x) + h(y + x) = g(y) + h(y) + g(-x) + h(x) \tag{9}$$

and replacing x, y with $-y$, $-x$ yields

$$g(-y + x) + h(-y - x) = g(-y) + h(-y) + g(x) + h(-x) \tag{10}$$

Subtracting (9) from (8) we get

$$g(x) - g(-x) = g(y) - g(-y) + g(x - y) - g(y - x) \tag{11}$$

and subtracting (10) from (8) we get

$$h(x + y) - h(-x - y) = h(x) - h(-x) + h(y) - h(-y). \tag{12}$$

From (11) we see that the function $G(x) = g(x) - g(-x)$ is additive. Since G is continuous, we have $G(x) = cx$ for some constant c. Now $G(1) = g(1) - g(-1) = 0$, so $c = 0$. Thus $g(x) = g(-x)$ for all x. In the same way, using (12), we conclude that $h(x) = h(-x)$ for all x. Now, taking $y = -x$ in (8) we get

$$g(2x) = 2g(x) + 2h(x)$$

and taking $y = x$ we get

$$h(2x) = 2g(x) + 2h(x).$$

It follows that $g = h$. Now (8) takes the form

$$g(x+y) + g(x-y) = 2g(x) + 2g(y), \text{ and } g(1) = 1. \tag{13}$$

We claim that the only continuous function which satisfies (13) is $g(x) = x^2$ (see an exercise below). Thus $F(x, y) = x^2 + y^2$.

Exercise. Let g be a function which satisfies condition (13).

a) Show that $g(m) = m^2$ for every integer m.

b) Show that $g(w) = w^2$ for every rational number whose denominator is a power of 2.

c) Suppose that g is continuous. Show that $g(x) = x^2$ for all x.

Exercise. Show that a continuous function F satisfies conditions (i) if and only if

$$F(x, y) = a(x^2 + y^2) + bx + cy + d$$

for some real numbers a, b, c, d.

Remark. Note that if F satisfies condition (i) and $h : \mathbb{R} \longrightarrow \mathbb{R}$ is any additive function, then $h \circ F$ also satisfies condition (i). Since there is a large supply of discontinuous additive functions, condition (i) itself is not sufficient to have the result from the exercise.

Problem. Is it true that a continuous function F such that F_0 and F_α have property $(*)$ for a single $\alpha \in (0, \pi/2)$ must be of the form $F(x, y) = a(x^2 + y^2) + bx + cy + d$ for some real numbers a, b, c, d?