Problem 1. A point P inside a convex quadrilateral $ABCD$ is such that the triangles ABP, BCP, CDP, ADP have all the same area. Prove that one of the diagonals halves the area of the quadrilateral.

Solution. We start by observing that it suffices to show that P is on one of the diagonals of $ABCD$. In fact, if P is on AC then the area of triangle ABC is the sum of areas of triangles ABP and BCP, and the area of triangle ADC is the sum of areas of triangles ADP and CDP. Thus the diagonal $\overline{AC}$ halves the area of our quadrilateral.

Let us also note that conversely, if the diagonal $\overline{AC}$ halves the area of our quadrilateral then taking for P the midpoint of $\overline{AC}$ we get a point satisfying the conditions of the problem.

First solution (after Josiah Moltz) Consider the line AP. Let B_1, D_1 be the perpendicular projections of B, D respectively on the line AP. The area of triangle ABP is $(AP \cdot BB_1)/2$ and the area of triangle ADP is $(AP \cdot DD_1)/2$. Since the triangles ABP and ADP have equal areas, we conclude that $BB_1 = DD_1$. Thus the segments $\overline{BB_1}$ and $\overline{DD_1}$ are parallel and have equal lengths. Consequently, $\overrightarrow{BB_1} = \overrightarrow{D_1D}$ (note that points B, D are on opposite sides of the line AP). It follows that either B, B_1, D, D_1 are collinear or BB_1DD_1 is a parallelogram. In the former case, we have $B_1 = D_1$ is the midpoint of $\overline{BD}$. In the latter case, the diagonals $\overline{BD}$ and $\overline{B_1D_1}$ intersect at the point which is the midpoint of both of them. In any case, the midpoint of $\overline{BD}$ is on the line AP.

The same argument applied to the projections of B, D on the line CP shows that the midpoint of $\overline{BD}$ is on the line CP. If the lines AP and CP are different, then P is the only point of intersection of the two lines, so P is the midpoint of $\overline{BD}$. If the lines AP and CP coincide, then P is on the diagonal AC.

Second solution. We start with the following useful observation. Consider an angle $\angle XOY$ and a number $k \geq 0$. For any point P let x_P, y_P be the distances from P to the lines OX, OY respectively. Then the collection of all points P inside the angle $\angle XOY$ such that $x_P = ky_P$ is a ray (i.e. half line) originating at O. We leave a proof of this fact as an exercise.

Returning to our problem, it follows from the above observation that the collection of all points Q inside the angle $\angle BAD$ such that the triangles ABQ and ADQ have equal area is the ray $AP^{\rightarrow}$. If M is the midpoint of $\overline{BD}$, then the triangles ABM and ADM have equal areas. Thus the midpoint of $\overline{BD}$ is on the line AP.

In the same way we show that the midpoint of $\overline{BD}$ is on the line CP. Now we proceed as in the first solution.

Third solution (after Gerald Marchesi). This solution uses the notion of cross product. Let u, v be two vectors in $\mathbb{R}^3$. If u, v are linearly dependent, i.e. one is a scalar multiple of the other, then the cross product $u \times v = 0$. Otherwise $u \times v$ is the unique vector w which is perpendicular to both u and v, whose length is equal to the area of the parallelogram spanned by u and v and whose direction is determined by the right-hand rule: if you put the index finger of your right hand along u and the middle finger along v then the thump points in the direction of w. Another way to define the direction of w is as follows: the plane determined by u and v has two sides and w points in the direction of the side which has the following property: when looking at the plane of u and v from that side the rotation of u towards w is counterclockwise.

The cross product has the following properties:

- $u \times v = -v \times u$, i.e. the cross product is skew-symmetric;
- $(u_1 + tu_2) \times v = u_1 \times v + tu_2 \times v$ for any scalar t, i.e. the cross product is bilinear.
- $u \times v = 0$ if and only if u, v are linearly dependent.

Consider now our quadrilateral $ABCD$ as embedded in $\mathbb{R}^3$ so that P is the origin (for example, we can assume that the plane in which $ABCD$ lies is the $x - y$ plane $z = 0$). Let $u_1 = \overrightarrow{PA}$, $u_2 = \overrightarrow{PB}$, $u_3 = \overrightarrow{PC}$, $u_4 = \overrightarrow{PD}$. The fact that the triangles PAB, PBC, PCD, PAD have equal areas implies that $u_1 \times u_2 = u_2 \times u_3 = u_3 \times u_4 = u_4 \times u_1$. Since $u_2 \times u_3 = -u_3 \times u_2$, we see that

$$0 = u_1 \times u_2 + u_3 \times u_2 = (u_1 + u_3) \times u_2.$$

Similarly

$$(u_1 + u_3) \times u_4 = 0.$$

If $u_1 + u_3 = 0$ then P is the midpoint of the diagonal $\overline{AC}$. Otherwise both u_2 and u_4 are scalar multiples of $u_1 + u_3$. This implies that B, P, D are collinear.

Fourth solution. There is no loss in generality if we assume that $P = (0, 0)$, $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$, $D = (d_1, d_2)$ and the vertices A, B, C, D are ordered counterclockwise. Then the areas of triangles PAB, PBC, PCD, PDA are $(a_1b_2 - a_2b_1)/2$, $(b_1c_2 - b_2c_1)/2$, $(c_1d_2 - c_2d_1)/2$, $(d_1a_2 - d_2a_1)/2$ respectively. Thus

$$a_1b_2 - a_2b_1 = b_1c_2 - b_2c_1 = c_1d_2 - c_2d_1 = d_1a_2 - d_2a_1.$$

It follows that $(a_1 + c_1)b_2 - (a_2 + c_2)b_1 = 0$ and $(a_1 + c_1)d_2 - (a_2 + c_2)d_1 = 0$. If $a_1 + c_1 = 0 = a_2 + c_2$ then P is the midpoint of $\overline{AC}$. Otherwise, points $(b_1, b_2) = B$ and $(d_1, d_2) = D$ both belong to the line $(a_1 + c_1)y - (a_2 + c_2)x = 0$. Point $P = (0, 0)$ is also on this line, so B, P, D are collinear.

Fifth solution. Let $\alpha, \beta, \gamma, \delta$ be the measures of the angles $\angle APB$, $\angle BPC$, $\angle CPD$, $\angle DPA$ respectively. Then $\alpha + \beta + \gamma + \delta = 2\pi$ and

$$\text{area}(\triangle ABP) = AP \cdot BP \cdot \sin\alpha, \quad \text{area}(\triangle BCP) = BP \cdot CP \cdot \sin\beta,$$

$$\text{area}(\triangle CDP) = CP \cdot DP \cdot \sin\gamma, \quad \text{area}(\triangle ADP) = AP \cdot DP \cdot \sin\delta = -AP \cdot DP \cdot \sin(\alpha + \beta + \gamma).$$

Since $\text{area}(\triangle ABP)\text{area}(\triangle CDP) = \text{area}(\triangle BCP)\text{area}(\triangle ADP)$, we see that

$$\sin\alpha \sin\gamma + \sin\beta \sin(\alpha + \beta + \gamma) = 0.$$

Recall that

$$\sin(\alpha + \beta + \gamma) = \sin(\alpha + \beta)\cos\gamma + \cos(\alpha + \beta)\sin\gamma$$

and

$$\sin\alpha = \sin(\alpha + \beta - \beta) = \sin(\alpha + \beta)\cos\beta - \cos(\alpha + \beta)\sin\beta.$$

Thus

$$0 = \sin\alpha \sin\gamma + \sin\beta \sin(\alpha + \beta + \gamma) =$$

$$\sin(\alpha + \beta)\cos\beta \sin\gamma - \cos(\alpha + \beta)\sin\beta \sin\gamma + \sin(\alpha + \beta)\sin\beta \cos\gamma + \cos(\alpha + \beta)\sin\beta \sin\gamma =$$

$$\sin(\alpha + \beta)(\sin\beta \cos\gamma + \cos\beta \sin\gamma) = \sin(\alpha + \beta)\sin(\beta + \gamma).$$

It follows that $\sin(\alpha + \beta) = 0$ or $\sin(\beta + \gamma) = 0$, i.e. $\alpha + \beta = \pi$ or $\beta + \gamma = \pi$. In the former case P is on the line AC and in the latter case P is on BD.

Sixth solution (after Matt Wolak). This solution requires some familiarity with affine transformations. Affine transformations take figures of equal areas to figures of equal areas. Thus if T is an affine transformation then our problem is true for a quadrilateral $ABCD$ and point P if and only if it is true for the quadrilateral $T(A)T(B)T(C)T(D)$ and the point $T(P)$. Also, for any two triangles there is an affine transformation mapping the first triangle onto the second. Thus, applying appropriate affine transformation, we may assume that $AB = AD$. Then P is equidistant from lines AB and AD, hence P is on the angle bisector of the angle $\angle BAD$ and the triangles BAP and DAP are congruent (side-angle-side). It follows that $BP = DP$ and the line AP bisects angle $\angle BPD$. Since triangles CBP and CDP have the same area, the point C is equidistant from the lines BP and CP, which easily implies that either the lines BP, CP coincide or C is on the angle bisector AP of the angle $\angle BPD$. In other words, either P is on BD or it is on AC.

Exercise. Prove the observation used in our second solution.

Exercise. Show that if the triangles ABP and ADP have equal areas and the triangles CBP and CDP have equal areas then either AC halves the area of the quadrilateral or P is the midpoint of $\overline{BD}$.

Exercise. Show that $\text{area}(\triangle ABP)\text{area}(\triangle CDP) = \text{area}(\triangle BCP)\text{area}(\triangle ADP)$ if and only if P is on one of the diagonals of $ABCD$.

Exercise. What can you say about $ABCD$ if not only the areas but also the perimeters of the triangles ABP, BCP, CDP, ADP coincide.