Problem 5. An investor in a casino is offered a choice of getting a return each time a certain game is played. The game is played by tossing N times a fair coin and recording the sequence of heads (H) and tails (T). Let h be the number of appearances of HH in the recorded sequence and let t be the number of appearances of HT. For example, when $N = 5$ and THHHT is recorded then $h = 2$ and $t = 1$. The investor can choose to either get h cents each time the game is played, or to get t cents each time the game is played. Which choice offers a better expected return?

Solution. Let Ω_N be the set of all sequence of heads (H) and tails (T) of length N. Clearly Ω_N has 2^N elements and each element can occur with probability $1/2^N$. Let X_{HT} be the random variable which to each sequence ω of Ω_N assigns the number of appearances of HT in ω. Similarly we define X_{HH}, X_{TT}, and X_{TH}. The problem asks us which of the random variable X_{HH} or X_{HT} has larger expectation. Recall that the expectation EX of a random variable X on Ω_N is defined as follows:

$$EX = \frac{1}{2^N} \sum_{\omega \in \Omega_N} X(\omega).$$

It should be obvious that for every ω we have the equality

$$X_{HT}(\omega) + X_{HH}(\omega) + X_{TT}(\omega) + X_{TH}(\omega) = N - 1$$

(a sequence of length N has $N - 1$ pairs of consecutive letters). It follows that

$$EX_{HT} + EX_{HH} + EX_{TT} + EX_{TH} = N - 1.$$

Define ω^* to be the sequence obtained from ω by replacing every T with H and every H with T. It is clear that $X_{HT}(\omega) = X_{TH}(\omega^*)$ and $X_{HH}(\omega) = X_{TT}(\omega^*)$. It follows that $EX_{HH} = EX_{TT}$ and $EX_{HT} = EX_{TH}$. Thus,

$$EX_{HT} + EX_{HH} = \frac{N-1}{2}.$$

We are now going to compute the number n_t of sequences ω such that $X_{HT}(\omega) = t$. Note that

$$EX_{HT} = \frac{1}{2^N} \sum_t n_t t.$$

Sequences ω such that $X_{HT}(\omega) = t$ can be build as follows. We start with listing HT t times. Before the first appearance of HT we insert the letter T a_0 times followed by the letter H inserted b_0 times (here a_0, b_0 are some non-negative integers). Between the i-th and $(i+1)$-st appearance of HT we insert a_i times the letter T followed by inserting b_i times the letter H. Finally, after the last appearance of HT we insert the letter T a_t times followed by the letter H inserted b_t times. The resulting sequence is in Ω_N if and only if $a_0 + b_0 + a_1 + b_1 + \ldots + a_t + b_t = N - 2t$. It follows that the number n_t is equal to the number of solutions of the equation

$$a_0 + b_0 + a_1 + b_1 + \ldots + a_t + b_t = N - 2t$$

in non-negative integers a_i, b_i, $i = 0, 1, \ldots t$. Equivalently, this is the same as the number of solutions of the equation

$$x_0 + y_0 + x_1 + y_1 + \ldots + x_t + y_t = N + 2$$

in positive integers x_i, y_i, $i = 0, 1, \ldots t$ (here $x_i = a_i + 1$ and $y_i = b_i + 1$).

We will use now the following result.

Theorem 1. The number of solutions of the equation $z_1 + z_2 + \ldots + z_m = M$ in positive integers $z_1, \ldots, z_m$ is equal to $\binom{M-1}{m-1}$.

Recall that $\binom{a}{b}$ is the binomial coefficient $\frac{a!}{b!(a-b)!}$, where $a \geq b \geq 0$. When $b > a$, we set $\binom{a}{b} = 0$.

We will first finish our solution using Theorem 1 and then we will provide a proof of the theorem. By Theorem 1 and our discussion above we see that

$$n_t = \binom{N+1}{2t+1}.$$

Clearly $\sum_t n_t$ counts all elements in Ω_N, i.e.

$$\sum_t \binom{N+1}{2t+1} = 2^N$$

(this can be proved in other ways, for example using the binomial formula applied to $(1-1)^{N+1} = 0$). It follows that

$$EX_{HT} = \frac{1}{2^N}\sum_t \binom{N+1}{2t+1} t = \frac{1}{2^{N+1}}\sum_t \binom{N+1}{2t+1}(2t+1) - \frac{1}{2^{N+1}}\sum_t \binom{N+1}{2t+1} = \frac{1}{2^{N+1}}\sum_t \binom{N+1}{2t+1}(2t+1) - \frac{1}{2}.$$

Note that

$$\binom{N+1}{2t+1}(2t+1) = (N+1)\binom{N}{2t}.$$

Thus

$$EX_{HT} = \frac{N+1}{2^{N+1}}\sum_t \binom{N}{2t} - \frac{1}{2} = \frac{N+1}{2^{N+1}}2^{N-1} - \frac{1}{2} = \frac{N-1}{4}.$$

We used here the fact that $\sum_t \binom{N}{2t} = 2^{N-1}$. This follows from our previous observation that $\sum_t \binom{N}{2t+1} = 2^{N-1}$ and the equality $\sum_t \binom{N}{t} = 2^N$.

We showed that $EX_{HT} = (N-1)/4$. Since we showed that $EX_{HT} + EX_{HH} = (N-1)/2$, we conclude that

$$EX_{HT} = EX_{HH} = \frac{N-1}{4}.$$

It remains to prove Theorem 1. Recall that $\binom{M-1}{m-1}$ is the number of subsets of size $m-1$ of the set $\{1, 2, \ldots, M-1\}$. Given a solution in positive integers $z_1, \ldots, z_m$ to the equation $z_1 + \ldots + z_m = M$ we get a subset $\{z_1, z_1+z_2, \ldots, z_1+z_2+\ldots+z_{m-1}\}$ of the set $\{1, 2, \ldots, M-1\}$. Conversely, given a subset $\{w_1, w_2, \ldots w_{m-1}\}$ of the set $\{1, 2, \ldots, M-1\}$, where $0 < w_1 < w_2 < \ldots < w_{m-1} < M$, we get a solution $z_1 = w_1, z_2 = w_2 - w_1, \ldots, z_{m-1} = w_{m-1} - w_{m-2}, z_m = M - w_{m-1}$ of the equation $z_1 + \ldots + z_m = M$ in positive integers. We established a bijection between subsets of size $m-1$ of the set $\{1, 2, \ldots, M-1\}$ and the solutions to the equation $z_1 + \ldots + z_m = M$ in positive integers. This clearly implies Theorem 1.

Second solution. For $\omega \in \Omega_N$ and $1 \le i \le N$ define

$$X_i(\omega) = \begin{cases} 1, & \text{if } \omega \text{ has H in the } i\text{-th place;} \\ 0, & \text{if } \omega \text{ has T in the } i\text{-th place.} \end{cases}$$

Note that

$$X_{HH}(\omega) = X_1(\omega)X_2(\omega) + X_2(\omega)X_3(\omega) + \ldots + X_{N-1}(\omega)X_N(\omega)$$

and

$$X_{HT}(\omega) = X_1(\omega)(1-X_2(\omega)) + X_2(\omega)(1-X_3(\omega)) + \ldots + X_{N-1}(\omega)(1-X_N(\omega)) = X_1(\omega) + \ldots + X_{N-1}(\omega) - X_{HH}(\omega).$$

It is easy to see that $EX_i = 1/2$ and that $X_1, \ldots, X_N$ are independent random variables. In particular, for $i \ne j$, we have $E(X_iX_j) = EX_i \cdot EX_j = 1/4$. It is now immediate that

$$EX_{HH} = E(X_1X_2) + E(X_2X_3) + \ldots + E(X_{N-1}X_N) = \frac{N-1}{4}$$

and

$$EX_{HT} = EX_1 + \ldots + EX_{N-1} - EX_{HH} = \frac{N-1}{2} - \frac{N-1}{4} = \frac{N-1}{4}.$$

Problem. What is more likely: $X_{HH} < X_{HT}$ or $X_{HH} > X_{HT}$?