Problem 5. For real numbers a, b, c consider the system of equations

$$x^2 + 2yz = a, \quad y^2 + 2xz = b, \quad z^2 + 2xy = c.$$

Prove that this system has at most one solution in real numbers x, y, z such that $x \geq y \geq z$ and $x + y + z \geq 0$. Prove that such a solution exists if and only if $a + b + c \geq 0$ and $b = \min(a, b, c)$. Here $\min(a, b, c)$ denotes the smallest number among a, b, c.

Solution. Suppose that x, y, z is a solution in real numbers to our system of equations. Adding the three equations we see that $(x + y + z)^2 = a + b + c$. Thus $a + b + c \geq 0$. Let $S = xy + xz + yz$. Adding the first two equations we get $x^2 + y^2 + 2yz + 2xz = a + b$ which is equivalent to $(x - y)^2 = a + b - 2S$. Similarly we see that $(y - z)^2 = b + c - 2S$ and $(x - z)^2 = a + c - 2S$. If $x \geq y \geq z$ then $x - z \geq x - y \geq 0$ and $x - z \geq y - z \geq 0$. It follows that $a + c - 2S \geq a + b - 2S$ and $a + c - 2S \geq b + c - 2S$, i.e. $c \geq b$ and $a \geq b$. In other words, $b = \min(a, b, c)$. We see that the condition that $a + b + c \geq 0$ and $b = \min(a, b, c)$ is necessary for having a solution in real numbers x, y, z such that $x \geq y \geq z$.

Suppose now that $a + b + c \geq 0$ and $b = \min(a, b, c)$. We need to show that there is unique solution in real numbers x, y, z such that $x + y + z \geq 0$ and $x \geq y \geq z$. Suppose that we have such a solution x, y, z. From our discussion above we know that

$$x - y = \sqrt{a + b - 2S}, \quad x - z = \sqrt{a + c - 2S}, \quad y - z = \sqrt{b + c - 2S}, \quad x + y + z = \sqrt{a + b + c}. \tag{1}$$

Thus

$$\sqrt{a + b - 2S} + \sqrt{b + c - 2S} = \sqrt{a + c - 2S}.$$

Squaring both sides, we get

$$a + c + 2b - 4S + 2\sqrt{(a + b - 2S)(b + c - 2S)} = a + c - 2S$$

i.e.

$$\sqrt{(a + b - 2S)(b + c - 2S)} = S - b.$$

Squaring again yields

$$3S^2 - 2S(a + b + c) + (ab + ac + bc) = 0.$$

We see that S is a solution to the quadratic equation $3t^2 - 2t(a + b + c) + (ab + ac + bc) = 0$. The discriminant of this equation is

$$4(a+b+c)^2 - 12(ab+ac+bc) = 4(a^2+b^2+c^2-ab-ac-bc) = 2((a-b)^2+(a-c)^2+(b-c)^2) = 4L,$$

where $L = ((a - b)^2 + (a - c)^2 + (b - c)^2)/2$ is non-negative. It follows that

$$S = \frac{(a + b + c) \pm \sqrt{L}}{3}.$$

However, from (1) we see that $a + b \geq 2S$, $a + c \geq 2S$, $b + c \geq 2S$, so $a + b + c \geq 3S$. Thus we must have

$$S = \frac{(a + b + c) - \sqrt{L}}{3}. \tag{2}$$

From (1) we see now that

$$3x = \sqrt{a + b + c} + \sqrt{a + b - 2S} + \sqrt{a + c - 2S} \tag{3}$$

$$3y = \sqrt{a + b + c} - \sqrt{a + b - 2S} + \sqrt{b + c - 2S} \tag{4}$$

$$3z = \sqrt{a + b + c} - \sqrt{a + c - 2S} - \sqrt{b + c - 2S}. \tag{5}$$

This proves that if the solution exists then it is unique and it is given by the formulas (2)-(5).

Conversely, define S by the formula (2). Note that

$$4L = (2c - a - b)^2 + 3(a - b)^2 = (2a - b - c)^2 + 3(b - c)^2 = (2b - a - c)^2 + 3(a - c)^2.$$

Thus $4L \geq (2c-a-b)^2$, so $2\sqrt{L} \geq 2c-a-b$ which implies that $a+b-2S \geq 0$. Similarly we see that $a+c-2S \geq 0$ and $b+c-2S \geq 0$. Since $b \leq a$ and $b \leq c$ we see that

$$(a-c)^2 = ((a-b)-(c-b))^2 \leq ((a-b)+(c-b))^2 = (2b-a-c)^2.$$

Thus $4L = (2b-a-c)^2 + 2(a-c)^2 \leq 4(2b-a-c)^2$ and therefore $\sqrt{L} \leq |2b-a-c| = a+c-2b$. This easily implies that $S \geq b$.

Now define x, y, z by the formulas (3)-(5). We will show that x, y, z are solutions to our system of equations. First note that x, y, z are real numbers and $x+y+z = \sqrt{a+b+c} \geq 0$. Since $b \leq a$ and $b \leq c$, we see that $x \geq y \geq z$. Starting with the equality (2) and going backwards in our reasoning above we see that $\sqrt{a+b-2S} + \sqrt{b+c-2S} = \sqrt{a+c-2S}$. This and formulas (3)-(5) imply that (1) holds for x, y, z. From (1) we see that

$$2(a+b+c) - 6S = (x-y)^2 + (x-z)^2 + (y-z)^2 = 2(x^2+y^2+z^2) - 2(xy+xz+yz) =$$

$$2(x+y+z)^2 - 6(xy+xz+yz) = 2(a+b+c) - 6(xy+xz+yz)$$

and therefore $S = xy+xz+yz$. Now

$$2a - 2S = (x-y)^2 + (x-z)^2 - (y-z)^2 = 2x^2 - 2xy - 2xz + 2yz = 2x^2 - 2S + 4yz$$

so $x^2 + 2yz = a$. Similarly we show that $y^2 + 2xz = b$ and $z^2 + 2xy = c$. This completes our argument that formulas (2)-(5) give the unique solution x, y, z to our system such that $x \geq y \geq z$ and $x+y+z \geq 0$.

Exercise. Suppose that $a+b+c \geq 0$ and $b = \min(a, b, c)$. Show that in addition to the unique solution x, y, z with $x+y+z \geq 0$ and $x \geq y \geq z$ our system of equations has solutions

$$-x, -y, -z; \quad \frac{-x+2y+2z}{3}, \frac{2x-y+2z}{3}, \frac{2x+2y-z}{3}; \quad \frac{x-2y-2z}{3}, \frac{-2x+y-2z}{3}, \frac{-2x-2y+z}{3},$$

and no other solutions.

Exercise. Verify the following identity:

$$3(xy+xz+yz)^2 - 2(xy+xz+yz)(x+y+z)^2 + (x^2+2yz)(y^2+2xz)(z^2+2xy) = 0.$$

Exercise. Study the solutions to our system in complex numbers (allowing a, b, c to be complex numbers too).