Problem 4. Find all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for any real numbers x, y either $f(x + f(y)) = f(x) + y$ or $f(f(x) + y) = x + f(y)$.

Solution. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be any functions such that for any real numbers x, y

$$f(x + f(y)) = f(x) + y \ \text{ or } \ f(f(x) + y) = x + f(y). \tag{1}$$

Then f has the following properties.

a) $f(x + f(x)) = x + f(x)$ for any x.
In, fact this follows from (1) by taking $x = y$.

b) f is one-to-one.
In fact, assume that $f(x) = f(y)$. Then, by (1), either

$$f(x) + y = f(x + f(y)) = f(x + f(x)) = x + f(x)$$

or

$$x + f(y) = f(f(x) + y) = f(f(y) + y) = f(y) + y.$$

Both cases imply that $x = y$.

c) $f(0) = 0$.
To see this, apply a) with $x = 0$ to get $f(f(0)) = f(0)$. Then use b) to conclude that $f(0) = 0$.

d) If $f(x) = x$ and $f(y) = y$ then $f(x + y) = x + y$.
Indeed, by (1), either

$$x + y = f(x) + y = f(x + f(y)) = f(x + y)$$

or

$$x + y = x + f(y) = f(f(x) + y) = f(x + y).$$

Both cases tell us that $f(x + y) = x + y$.

e) If $f(x) = x$ then $f(-x) = -x$.
Indeed, using c) and taking $y = -x$ in (1) we get either

$$f(x + f(-x)) = f(x) + (-x) = x - x = 0 = f(0)$$

or

$$0 = f(x - x) = f(f(x) - x) = x + f(-x).$$

In the former case, $x + f(-x) = 0$ by b). Thus either case implies that $f(-x) = -x$.

Let $F = \{x : f(x) = x\}$ be the set of fixed points of f. Then $0 \in F$ by c). By d) and e), the set F is closed under addition and subtraction (this means that F is a group under addition). In particular, if $a \in F$ then $ma \in F$ for any integer m.

Suppose now that f is continuous. Let $g(x) = f(x) + x$. Then $g(0) = 0$ and g is continuous. If g is constant then $g(x) = 0$ for all x and therefore $f(x) = -x$ for all x. If g is not constant, then $g(a) \neq 0$ for some a. Note that by a) all values of g belong to F. By the intermediate value theorem, all numbers between 0 and $g(a)$ belong to F. Thus the interval $[0, |g(a)|]$ is contained in F (by e)). Now if x is any real number than there is an integer m such that $x/m \in [0, |f(a)|]$. Thus $x/m \in F$ and therefore $x = m(x/m) \in F$. In other words, $F = \mathbb{R}$, i.e. $f(x) = x$ for all x.

Exercise. Show that f has the following additional properties:

f) If $a \in F$ then $a - x = f(a - f(x))$ for all x.

g) If $a \in F$ then $f(a + x) = a + f(x)$ for all x.

Exercise. Show that the function $g(x) = x + f(x)$ satisfies the conditions of Problem 2. Conclude that $f(x) + f(-x)$ assumes at most two different values. This provides a simple solution to Problem 6 from the 65th International Mathematical Olympiad (I discovered problem 2 in order to solve the IMO problem).