Problem 3. Three players are playing a game. They are taking turns placing kings on a 1000x1000 chessboard, so that the newly-placed king is not adjacent (directly or diagonally) to any of the previously placed kings (i.e. the kings are in non-attacking positions). Whoever cannot place a king, loses. Prove that if any two of the players cooperate, they can make the third player lose.

Solution. Let us start with several conventions and simple observations.

1. Our game can be played on any board. By a board we mean here any subset of the infinite chessboard which consists of finitely many of its 1×1 squares. We call the player who goes first player 1, we call the player who goes second player 2 and we call the player who goes third player 3.

2. After several moves are made, we can consider the remaining game as a new game, played on a new board which is obtained from the original board by removing from it all squares touching a square occupied by a king (including the occupied squares). Note that in this new game the roles of the players may change (i.e. player 1 in the original game may become player 2 or player 3 in the new game).

3. For the game on any board, at least one of the 3 players does not have a non-losing strategy. In fact, if each of the 3 players had a non-losing strategy, then the game would never end. This means that there is a player, call her A, such that the other 2 players have a strategy to make A lose. (This is similar to the important fact that in any 2 player finite game which cannot end in a draw, one of the players has a winning strategy.) This observation prompts the following terminology. We call a board a **1-2 board** if players 1 and 2 can make player 3 lose. Similarly, a board is a **1-3 board** if players 1 and 3 have a strategy to make player 2 lose. Finally, a **2-3 board** is a board such that players 2 and 3 have a strategy to make player 1 lose. Our key observation says that every board is of one of these 3 types. The problem asks us to show that a 1000×1000 chessboard is of all 3 types.

4. We call two boards **disjoint** if no square of one of the boards touches a square of the other board. If a board consists of two disjoint pieces, then a move made on one of the pieces has no impact on the other piece.

Some examples:
a) Consider a board which consists of n pairwise disjoint 1×1 squares. There are no constraints in placing kings on such a board. Thus, if 3 divides n then player 1 will lose no matter how the game is played. It follows that this board is a 2-3 board but not a 1-2 or 1-3 board. Similarly, when 3 divides $n-1$, it is a 1-3 board but not a 1-2 or 2-3 board and when 3 divides $n-2$, it is a 1-2 board but not a 1-3 or 2-3 board

b) Consider now a board which is a 3×1 rectangle. It is a 1-2 board. In fact, player 1 can put a king in a corner, then player 2 can put a king in the second corner and player 3 loses. It is also a 1-3 board: player 1 places a king in the middle square and then player 2 has no move (loses). Clearly it is not a 2-3 board.

c) Consider now a board which consists of two disjoint 3×1 rectangles. It is a 1-2 board: if player 1 places a king in the middle of one of the rectangles and player 2 places a king in the middle of the other rectangle, then player 3 loses. It is also a 2-3 board: if player 1 starts by placing a king in a corner of one of the rectangles, then player 2 can place a king in the other corner of the same rectangle and player 3 can place a king in the center of the second rectangle and then player 1 loses; if player 1 starts by placing a king in the center of one of the rectangles then player 2 places a king in a corner of the second rectangle and player 3 places a king in the second corner of that rectangle; player 1 loses again. We leave it as a simple exercise to show that it is not a 1-3 board.

d) Consider now a board which consists of three pairwise disjoint 3×1 rectangles. We claim that this board is of all three types. Indeed, it is a 1-2 board: if player 1 places a king in a corner of one of the rectangles and player 2 places a king in the other corner of the same rectangle, then the remaining game is played on a board from Example c) with player 3 going first. Thus players 1 and 2 have a strategy to make player 3 lose (as the game is now on a 2-3 board).
It is also a 1-3 board: if player 1 places a king in the middle of one of the rectangles, then the remaining game is played on a board from Example c) with player 2 going first. Thus players 1 and 3 have a strategy to make player 2 lose.
Finally, it is a 2-3 board: if player 1 starts by placing a king in the middle of one of the rectangles, then

the remaining game is played on a board from Example c) with player 2 going first and player 3 going second. Since the board is a 1-2 board, players 2 and 3 have a strategy to make player 1 lose. If player 1 starts by placing a king in a corner of one of the rectangles then players 2 and 3 place kings in corners of the remaining two rectangles. At this point the remaining game is on a board from Example a) with player 1 going first and $n = 3$. Thus player 1 loses again.

Consider now a board B which is a union of two disjoint boards B_1, B_2.

5. **Suppose that both boards B_1, B_2 are 1-3 boards. Then B is a 1-2 board.** In fact, player 1 starts by playing on B_1 according to the 1-3 strategy for B_1. Player 2 makes first move on the board B_2 according to 1-3 strategy for B_2 (so player 2 plays as player 1 on the board B_2). Now each time player 3 moves on B_1 (she will be the player 2 for both the game played on B_1 and B_2), player 2 and player 1 move according to the 1-3 strategy for B_1; if player 2 moves on B_2, player 1 and player 2 move on B_2 according to the 1-3 strategy for B_2. This will guarantee that player 3 loses on each board.

6. **If B_1 is a 1-2 board and B_2 is a 2-3 board then B is a 1-2 board.** In fact, players 1 and 2 start by playing on B_1 according to the 1-2 strategy for B_1. Now each time player 3 moves on B_1 (she will still be player 3 on B_1), player 1 and player 2 move according to the 1-2 strategy for B_1; if player 3 moves on B_2 (she will be player 1 on B_2), player 1 and player 2 move on B_2 according to the 2-3 strategy for B_2. This will guarantee that player 3 loses on each board.

7. Combining 5 and 6 we get the following key conclusion: **if one of the boards B_1 and B_2 is of all three types, then B is a 1-2 board.**

Exercise. Show that

(i) if B_1 is a 1-3 board and B_2 is a 2-3 board then B is a 1-3 board.

(ii) if both B_1 and B_2 are 2-3 boards then so is B.

We are now ready to focus on solving our problem. Applying obesrvation 7. to a board B with B_1 being the board from example d) we see that any such board is a 1-2 board. This suggest the following strategy: the two cooperating players, call them A and B, should find a way to play so that at some point after the third player moves (call her C) the remaining game is played on a board which consists of two disjoint pieces, one of which is the board from example d). Since such a board is of type 1-2 and player C becomes player 3 in the remaining game, players A and B have a strategyto make the player C lose.

In order to finish the solution we need to show that the two cooperating players can always achieve a position outlined in the above strategy. We will call the 1×1 square in the i-th row and j-th column of the 1000×1000 chessboard the (i, j) square of the board. Denote by R_i ($i = 0, 1, \ldots, 89$ the 11×5 rectangular sub-boards of the 1000×1000 chessboard made by the first 5 columns and rows $11i+1, 11i+2, \ldots 11i+11$. In other words, R_i consists of squares (a, b) such that $11i+1 \leq a \leq 11i+11$ and $1 \leq b \leq 5$. Let B_i be the 3×1 rectangle consisting of squares $(11i + 5, 1), (11i + 6, 1), (11i + 7, 1)$. If at some point of the game we have kings atsquares $(11i + 6, 3)$, $(11i + 3, 2)$ and $(11i + 9, 2)$ and no other king in R_i then the remaining game is played on a board which splits into two disjoint pieces one of which is B_i. We will say in this situation that R_i is **well positioned**. Thus if at some point of the game after C moves we have 3 well positioned rectangles R_k, R_l, R_m then the remaining game is played on a 1-2 board and therefore players A and B have a way to make the player C lose. Here is how players A and B create 3 well positioned rectangles. Until player C's 30-th move each of A and B choses an i such that R_i has no kings in it and places a king in square $(11i + 6, 3)$. This can be done since we have 90 rectangles to choose from. After player C's 30th move we will have 58 rectangles R_i with a king in square $(11i + 6, 3)$. At most 30 of the 58 rectangles can contain another king, since C made 30 moves. So we have 28 rectangles $R_{m_1}, \ldots, R_{m_{28}}$ which contain only one king, and the king in R_{m_i} is in square $(11m_i + 6, 3)$. Both players A and B in their next 9 moves (after player C's 30th move) pick rectangle R_{m_i} which has only one king and places a king in square $(11m_i + 3, 2)$. This is always possible since we have 28 rectangles to pick from. Thus after player C's 39th move we will have 18 of the rectangles R_{m_i} with kings in squares $(11m_i+6, 3)$, $(11m_i+3, 2)$. At most nine of these rectangles can have another king (as player C made only 9 moves). Thus we have 9 rectangles $R_{n_1}, \ldots, R_{n_9}$ such that R_{n_i} has kings in squares $(11n_i + 6, 3)$, $(11n_i + 3, 2)$ and no other kings in it. Now players A and B in their next 3 moves (after player C's 39th move) pick n_i such that R_{n_i} has kings in squares $(11n_i + 6, 3)$, $(11n_i + 3, 2)$ and no ther kings and place a king in square $(11n_i + 9, 2)$. This is always possible since we have 9 rectangles to pick from. Thus, after player C's 42nd move, we have six of the rectangles R_{n_i} which have kings in squares $(11n_i + 6, 3)$, $(11n_i + 3, 2)$, $(11n_i + 9, 2)$ and at most 3 of them can have another king (since C made only 3 moves). This means that at least 3 of the 6 rectangles are well positioned. As we observed before, this means that players A, B have a strategy to continue playing so that C loses.

Remark. It is not hard to see that the strategy to create 3 well positioned 11×3 rectangles was not the most economical one and it could be adjusted to work for $m \times n$ boards with m, n a bit smaller than 1000. I do not know the answers to the following questions.

Problem 1. Find all $m \times n$ boards which are of all 3 types.

Problem 2. Which of the 3 types is the 8×8 board?

Problem 3. Consider the 2 player version of the game. Which of the 2 players has a winning strategy on an $n \times n$ board?

It is not hard to see that when n is odd then the player who goes first has a winning strategy. For $n = 4$ the player who goes second has a winning strategy (in fact wins regardless of how the game is played) and for $n = 6$ the player who goes first has a winning strategy. I do not know the answer for the 8×8 chessboard.