Problem 1. Find all natural numbers $n > 1$ such that $2! + 3! + \ldots + n!$ is a cube of an integer.

Solution. Our solution is based on the following useful observation.

Proposition. Any cube of an integer yields reminder 0, 1, or 6 when divided by 7. In other words, for any integer n one of the numbers n^3, $n^3 - 1$, $n^3 + 1 = (n^3 - 6) + 7$ is divisible by 7.

One way to justify this is to use Fermat's Little Theorem: if p is a prime number then $n^p - n$ is divisible by p for any integer n. Applying this to the case $p = 7$ we see that 7 divides $n^7 - n = n(n^3 - 1)(n^3 + 1)$, and consequently one of the numbers n^3, $n^3 - 1$, $n^3 + 1$ is divisible by 7. One can also check directly that each of the numbers $0^3, 1^3, \ldots, 6^3$ is congruent to 0 or ± 1 modulo 7, which easily implies the proposition.

We verify by direct computation that for $n \leq 5$ the number $2! + \ldots + n!$ is a cube if and only if $n = 3$. Note that $2! + 3! + 4! + 5! + 6! = 872 = 4 + 7 \cdot 124$. For any $k \geq 7$ the number $k!$ is divisible by 7. It follows that for $n \geq 6$ we have $2! + \ldots + n! = 4 + 7m$ for some integer m. In other words, $2! + \ldots + n!$ yields reminder 4 when divided by 7. It follows from the Proposition that $2! + \ldots + n!$ is not a cube of an integer when $n \geq 6$. Thus $n = 3$ is the only solution.

Remark. It is natural to ask whether $2! + \ldots + n!$ can be an m-th power of some integer for $m > 1$. Note that $k!$ is divisible by 16 for $k \geq 6$. Thus, for $n \geq 5$ we have

$$2! + \ldots + n! = 2! + \ldots + 5! + 16M = 152 + 16M = 8(19 + 2M)$$

for some integer M. Since $19 + 2M$ is odd, we see that 2^3 is the highest power of 2 which divides $2! + \ldots + n!$. If $2! + \ldots + n! = a^m$ for some $m > 1$ and the highest power of 2 which divides a is 2^u then the highest power of 2 which divides a^m is 2^{mu}. This means that $mu = 3$, so $m = 3$. But we proved that $2! + \ldots + n!$ is not a cube for $n \geq 5$. Thus, when $n \geq 5$, the number $2! + \ldots + n!$ is not an m-th power for any $m > 1$. Note that $2! + 3! + 4! = 32 = 2^5$ and $2! + 3! = 8 = 2^3$.

Exercise. Find all $n > 1$ for which the number $1! + 2! + \ldots + n!$ is a k-th power for some $k > 1$.