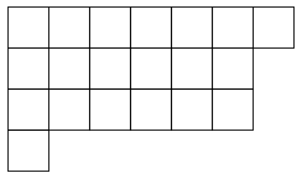
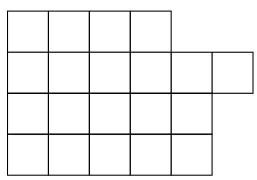
Problem 7. 5050 bats live in groups occupying some of the 5050 caves in the Magic Mountains (where bats live forever). Every night one bat from each occupied cave flies out. The bats meet and play together all night. At dawn they all go to one (the same) of the unoccupied caves to rest. Show that from some point on there will be the same number of bats out every night.

Solution 1 (after Slava Kargin). Let us suppose that on some day the bats are in k caves for some $1 \le k \le 5050$. Suppose the numbers of bats in occupied caves are $a_1, \ldots, a_k$, where $a_1 \ge a_2 \ge \ldots \ge a_k$. Clearly $a_1 + \ldots + a_k = 5050$. We will represent this by a diagram build with 5050 unit squares. The diagram consists of k left-justified rows of a_i unit squares stacked in decreasing order (so the i-th row consists of a_i unit squares). For example, if we had 20 bats in four caves ($k = 4$) with $7, 6, 6, 1$ bats respectively, the corresponding diagram would look as follows:

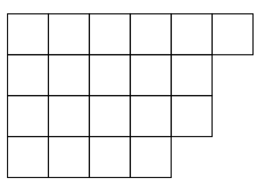


Such diagrams are often called **Young diagrams**. We will keep track of all the unit squares by assigning to the square in i-th row and j-th column its "coordinates" (i, j). The **level** of a unit square is the sum of its coordinates. So the square in the upper left corner has level 2, and so on.

During the night there will be k bats out and next day the numbers of bats in caves will be $a_1 - 1, a_2 - 1, \ldots, a_k - 1$ and k (disregard all 0's in this sequence). We can construct the diagram representing the configuration of bats on the next day in two steps as follows: in the first step we cut off the first column and place it on top of the first row to make a new diagram. Applying this to the diagram in the above example we get:



It is easy to see that the square with coordinates $(i, 1)$ in the original diagram now has coordinates $(1, i)$ and the square with coordinates (i, j), $j \ge 2$ in the original diagram now has coordinates $(i + 1, j - 1)$. In the second step, we move each of the columns $k + 1, k + 2, \ldots$ one unit up. In our example, we lift columns 5 and 6 one unit up and get:



The square with coordinates (i, j), $j > k$ in the diagram after the first step will now have coordinates $(i - 1, j)$ and for $j \le k$ the coordinates do not change. These two steps, when applied to the diagram representing the distribution of bats on a given day, produce the diagram for the next day. In terms of coordinates, we can describe this process as follows: the square with coordinates (i, j) in the diagram for a given day is moved on the next day to the square whose coordinates are given by the following formula

$$\begin{cases} (1, i) & \text{if j=1} \\ (i+1, j-1) & \text{if } 1 < j \le k+1 \\ (i, j-1) & \text{if } j > k+1. \end{cases}$$

Applying this to every square of the diagram, we obtain the diagram for the next day. It is straightforward to see that the level of any square can not increase when it is moved according to the above rule. This means that the sum of the levels of all the squares in the diagram for any day is smaller or equal than the analogous sum on a previous day. In other words, the sum of the levels of all squares in a diagram is a non-decreasing function of the day. Any non-decreasing sequence of positive integers must be eventually constant. Thus from some day N on the levels of each square remain constant. This means that squares of level l move from day N on in a cycle of length $l-1$ along its level as follows:

$$(i, l-i) \mapsto (i+1, l-i-1) \mapsto \ldots \mapsto (l-1, 1) \mapsto (1, l-1) \mapsto (2, l-2) \mapsto \ldots \mapsto (i, l-i) \mapsto \ldots.$$

Suppose that on day N some square of level l is not in the diagram for that day, say square $(i, l-i)$ is not there. We claim that the diagram has no squares of level larger than l. It is clear that if a Young diagram has no squares of some level then it has no squares of any larger level. Thus, if our claim were false, the diagram would have a square $(b, l+1-b)$ of level $l+1$. Note that then the diagram on day $N+l-i$ is missing the square $(1, l-1)$. The same is true for any day of the form $N+l-i+m(l-1)$ for any non-negative integer m (as the squares of level l move in a cycle of length $l-1$). Similarly, the diagram on any day of the form $N+l+1-b+nl$ contains the square $(1, l)$. If we can find non-negative integers m, n such that $M = N+l-i+m(l-1) = N+l+1-b+nl$ then the diagram on day M contains the square $(1, l)$ and is missing the square $(1, l-1)$. This is clearly impossible (as each row of a Young diagram is a one piece). The contradiction justifies our claim that the diagram on day N has no squares of level larger than l. It remains to show that such m, n indeed exist, i.e. that $nl - m(l-1) = b-i-1$. If $b-i-1 \geq 0$ we may take $m = n = b-i-1$. If $b-i-1 < 0$ take $n = (l-2)(i+1-b)$ and $m = (l-1)(i+1-b)$.

We have shown that there is $l \geq 2$ such that the diagram on day N contains all squares of level less than l, some squares of level l, and no squares of any higher level. Since there are $t-1$ squares of level t, our diagram contains $1+2+\ldots+(l-1)+s$ squares for some $0 < s \leq l$. Thus $1+2+\ldots+(l-1)+s = 5050 = 1+2+\ldots+100$. It is straightforward to see that this implies that $l = 100 = s$. This means that the diagram on day N consists of 100 rows of lengths $100, 99, 98, \ldots, 1$. The same is then true on any day after the N-th day. Thus from day N on there will be 100 bats out every night.

Second solution. Let us declare one particular day as day 0 (the day we start observing the bats). Let f_n be the number of bats out on the n-th night. This means that on the n-th day we have f_n occupied caves. Note that any occupied cave is loosing one bat every night until it becomes empty. It follows that if $n \geq 5050$ then any group of bats living on day n in the same cave was created after the k-th night for some $n > k \geq n-5050$ and on the $(k+1)$-st day it consisted of f_k bats. This group has been loosing one bat every night so on day n is has $f_k - (n-k) + 1$ members. Thus $f_k - (n-k) + 1 \geq 1$, i.e. $f_k + k \geq n$. Conversely, if $k < n$ and $f_k + k \geq n$ then the f_k bats out on the k-th night form a group living in the same cave on day $k+1$ and this group on day n has $f_k - (n-k) + 1 > 0$ bats. To summarize this discussion, for $n > 5050$ we have

$$f_n = |\{k < n : f_k + k \geq n\}|$$

and

$$5050 = \sum_k (f_k - (n-k) + 1), \tag{1}$$

where the sum is over all $k < n$ such that $f_k + k \geq n$. This should immediately bring into the consideration Problem 6 from this semester. We can not apply Problem 6 yet, since the sequence f_n is only defined for non-negative n. However, consider the sequence $s_n = (f_n, f_{n+1}, \ldots, f_{n+5050})$. Since this sequence can assume only finitely many different values, there exist $0 < K < M$ such that $s_K = s_M$. This means that $f_{K+i} = f_{M+i}$ for $i = 0, \ldots, 5050$. We claim that this implies that $f_{K+i} = f_{M+i}$ for every $i \geq 0$. Indeed, we already know this for $i \leq 5050$ and if this holds for all $i < n$ for some $n > 5050$ then

$$f_{K+n} = |\{k < K+n : f_k + k \geq K+n\}| = |\{K \leq k < K+n : f_k + k \geq K+n\}|$$
$$= \{0 \leq i < n : f_{K+i} + K + i \geq K+n\}| = |\{0 \leq i < n : f_{M+i} + M + i \geq M+n\}|$$
$$= |\{k < M+n : f_k + k \geq M+n\}| = f_{M+n}$$

so the claim holds for $i = n$. Thus our claim is true by induction. It follows that if $t = M-K$ then $f_{k+t} = f_k$ for every $k \geq K$. In other words, the sequence (f_k) is periodic of period t for $k \geq K$. Define $d_k = f_k$ for $k \geq K$ and extend this sequence to a periodic sequence (d_k) defined for all integers k. Then the numbers d_k have the property that $d_k = |\{i < k : d_i + i \geq k\}|$ for every $k \in \mathbb{Z}$. By Problem 6, $d_k \in \{n-1, n\}$ and $d_{k+n} = d_k$ for all k, where n is the largest value of the sequence d_k. Consider any k

for which $d_k = n$ and let $D_k = \{i < k : d_i + i \geq k\}$. Since $d_i \leq n$, any i in the set D_k must be at least $k - n$, so $D_k \subseteq \{k-n, k-n+1, \ldots, k-1\}$. On the other hand, D_k has $d_k = n$ elements. It follows that $D_k = \{k-n, \ldots, k-1\}$. By (1),

$$5050 = \sum_{i \in D_k} (d_i - (k-i) + 1 = \sum_{i=k-n}^{k-1} (d_i + i + 1 - k) = \sum_{i=k-n}^{k-1} d_i + \sum_{i=k-n}^{k-1} (i + 1 - k) = \sum_{i=k-n}^{k-1} d_i - \frac{(n-1)n}{2}$$

Recall that $n \geq d_i \geq n-1$ for all i and $d_{k-n} = d_k = n$. Suppose that not all d_i are equal to n. Then

$$n(n-1) < \sum_{i=k-n}^{k-1} d_i < n^2.$$

and therefore

$$n(n-1) - \frac{n(n-1)}{2} < \sum_{i=k-n}^{k-1} d_i - \frac{n(n-1)}{2} = 5050 < n^2 - \frac{n(n-1)}{2},$$

i.e.

$$\frac{n(n-1)}{2} < 5050 < \frac{n(n+1)}{2}.$$

In other words,

$$1 + 2 + \ldots + (n-1) < 1 + 2 + \ldots + 100 < 1 + 2 + \ldots + n$$

which is clearly not possible. This proves that $d_i = n = 100$ for $i = k-n, \ldots, k-1$ and therefore $d_i = 100$ for all i (recall that $d_{i+n} = d_i$). Thus $f_k = d_k = 100$ for all $k \geq K$. This means than from day K on we have 100 bats out every night.

Problem. Suppose we have 5050 coins divided into several stacks. Pick all coins from a largest stack and keep adding them to the other stacks, starting from largest down and putting one coin to each stack until possible (so if you have more coins than stacks, you create some new stacks with one coin at the end). Prove that from some point on the largest stack will always have the same number of coins.