Problem 6. Non-negative integers d_i, $i \in \mathbb{Z}$, satisfy the following condition:

$$d_k = |\{i < k : d_i + i \geq k\}|$$

for every integer k. Prove that there is a positive integer n such that

$$d_k \in \{n-1, n\} \quad \text{and} \quad d_{k+n} = d_k$$

for every integer k.

Solution. Let $D_k = \{i < k : d_i + i \geq k\}$. According to the definition of d_k, for every integer k the set D_k is finite with exactly d_k elements.

Suppose that $d_k = 0$ for some k. Then the set D_k is empty, hence $d_i + i < k$ for all $i < k$. In particular, $d_{k-1} + (k-1) < k$, i.e. $d_{k-1} < 1$. Thus $d_{k-1} = 0$. Note that $d_k + k = k < k+1$. It follows that $d_i + i < k+1$ for every $i < k+1$. In other words, the set D_{k+1} is empty and $d_{k+1} = 0$. By obvious induction we see that $d_i = 0$ for all integers i. We see that if $d_k = 0$ for some k then all terms d_i are 0 and the conclusion of the problem is true with $n = 1$.

From now on we assume that $d_k > 0$ for all k. Let $B_k = \{i : d_i + i = k\}$. Clearly $B_k \subseteq D_k$. Note that for $i < k$ we have $i + d_i \geq k+1$ if and only if $i + d_i \geq k$ and $i + d_i \neq k$. In other words, $D_{k+1} - \{k\} = D_k - B_k$. Since $d_k + k \geq 1 + k$, we have $k \in D_{k+1}$ and therefore

$$D_{k+1} = \{k\} \cup D_k - B_k \tag{1}$$

and consequently

$$d_{k+1} = 1 + d_k - |B_k| \leq 1 + d_k. \tag{2}$$

From (2) and obvious induction, we see that

Lemma 1. If $k \geq l$ then $d_k \leq d_l + k - l$.

Let m be the smallest among the positive integrs d_i, $i \in \mathbb{Z}$. Note that $d_{k-i} + (k-i) \geq m + k - i \geq k$ for $i = 1, \ldots, m$ and any integer k. Thus $\{k-1, \ldots, k-m\} \subseteq D_k$. Suppose now that $d_k = m$ for some k. Then we must have $\{k-1, \ldots, k-m\} = D_k$. It follows that $k - m - 1 \notin D_k$ and therefore $d_{k-(m+1)} + k - (m+1) < k$, i.e. $d_{k-(m+1)} < m+1$. We conclude that $d_{k-(m+1)} = m$. By obvious induction we get

Lemma 2. If $d_k = m$ then $d_{k-t(m+1)} = m$ for all non-negative integers t.

For any integer k define $N_k = \sum_{i \in D_k} (d_i + i + 1 - k)$. By (1) we have

$$N_{k+1} = (d_k + k + 1 - (k+1)) + \sum_{i \in D_k} (d_i + i + 1 - (k+1)) - \sum_{i \in B_k} (d_i + i + 1 - (k+1))$$

$$= d_k + \sum_{i \in D_k} (d_i + i + 1 - k) - \sum_{i \in D_k} 1 = d_k + N_k - d_k = N_k.$$

It follows that the sequence N_k is constant, i.e.

Lemma 3. $\sum_{i \in D_k} (d_i + i + 1 - k) = N$ for some N and all integers k.

Note that for $i \in D_k$ we have $i + d_i + 1 - k \geq k + 1 - k = 1$, so $N \geq \sum_{i \in D_k} 1 = d_k$. Thus the sequence d_k, $k \in \mathbb{Z}$ is bounded above. Let M be the largest among the positive integers d_i, $i \in \mathbb{Z}$. If $i < k - M$ then $d_i + i < M + k - M = k$. It follows that $D_k \subseteq \{k-1, \ldots, k-M\}$. In particular, $d_k = M$ if and only if $D_k = \{k-1, \ldots, k-M\}$. We see that if $d_k = M$ then $d_{k-M} + k - M \geq k$, i.e. $d_{k-M} \geq M$. Thus $d_{k-M} = M$ and an obvious induction yields

Lemma 4. If $d_k = M$ then $d_{k-tM} = M$ for all non-negative integers t.

Now we can prove the key observation:

Lemma 5. $M \leq m + 1$.

Proof. Suppose that $M > m+1$. There exist a, b such that $d_a = m$ and $D_b = M$. Among all non-negative integers s, t choose a pair s, t such that the number $(b - sM) - (a - t(m+1))$ is non-negative and smallest possible. If $(b - sM) - (a - t(m+1)) \geq M - (m+1)$ then $(b - sM) - (a - t(m+1)) > (b - (s+1)M) - (a - (t+1)(m+1)) = (b - sM) - (a - t(m+1)) - (M - (m+1)) \geq 0$, contrary to our choice of s, t. Thus we have $M - (m+1) > (b - sM) - (a - t(m+1)) \geq > 0$. Set $k = b - sM$ and $l = a - t(m+1)$. Then $M - m > k - l \geq 0$. By Lemma 2 and Lemma 4 we have $d_k = M$ and $d_l = m$. Lemma 1 tells us that $M - m = d_k - d_l \leq k - l$, which contradicts the fact that $M - m > k - l$. This contradiction shows that we must have $M \leq m+1$.

From lemma 5 we see that either $M = m$ or $M = m+1$. If $M = m$ then the sequence d_k is constant that the conclusion of our problem holds with $n = m+1$. If $M = m+1$ then $d_k \in \{m, m+1\}$ for all k. Take $n = m+1$. If $d_k = m$ then $d_{k-n} = m$ by Lemma 2. If $d_k = M = m+1$ then $d_{k-n} = M$ by Lemma 4. We see that $d_{k-n} = d_k$ for all k. Thus we showed that $d_k \in \{n-1, n\}$ and $d_{k+n} = d_k$ for all k, as required.

Second solution (after Slava Kargin): We start with Lemma 1 and Lemma 2 as in the first solution. We will show that $d_k \leq m+1$ for all k. Suppose that there is a such that $d_a > m+1$. There is b such that $d_b = m$. By Lemma 2, we may assume that $b < a$. Let $k < a$ be largest such that $d_k = m$. Then, using (2), we see that there is $t > 0$ such that $d_{k+1} = \ldots = d_{k+t} = m+1$ and $d_{k+t+1} = m+2$. We may assume that t is smallest possible. This means that if $d_p = m$ and $d_q > m+1$ for some $p < q$ then $q - p \geq t+1$. From $d_k = m$ we conclude $D_k = \{k-1, \ldots, k-m\}$. Recall now that $D_{j+1} \subseteq D_j \cup \{j\}$ for all j. It follows that $D_{k+i} \subseteq \{k-m, k-m+1, \ldots, k+i-1\}$ for $i \geq 0$. Since D_{k+t+1} has $m+2$ elements, $k+t-a \in D_{k+t+1}$ for some $a \geq m+1$. Thus $d_{k+t-a} + k+t-a \geq k+t+1$, i.e. $d_{k+t-a} \geq a+1 \geq m+2$. We have $k+t-a \geq k-m$ and $d_{k-m-1} = m$ (Lemma 2) and $d_{k+t-a} \geq m+2$. Our choice of t implies that $k+t-a-(k-m-1) \geq t+1$. This however means that $m \geq a$, which contradicts the inequality $a \geq m+1$. The contradiction shows that we must have $d_k \leq m+1$ for all k. In other words, $d_k \in \{m, m+1\}$ for all k. The fact that $d_{k+m+1} = d_k$ can be now established as in the first solution.