Problem 5. Consider a set S of n distinct points on a plane. A circle is called minimal for S if every point of S is either on the circle or inside the circle and there are at lest 3 points from S on the circle. What is the largest possible number of minimal circles a set with n points can have?

Solution. Let $M(n)$ denote the number in question. Clearly a set with one or two points has no minimal circles, so $M(1) = M(2) = 0$. If S consists of three non-collinear points A, B, C then S has unique minimal circle, namely the circumcircle of the triangle ABC. If A, B, C are collinear, then S has no minimal circles. Thus $M(3) = 1$. We will prove that $M(n) = n - 2$.

To proceed, we need some basic facts about convex sets. Recall that a subset P of a plane is called **convex** if for any two points A, B in P the whole segment $\overline{AB}$ is contained in P. Examples of convex sets: a segment, a line, each side of a line (every line divides the plane into two pieces, called sides of the line), interior of any circle, any disc (i.e. a circle together with its interior). Intersection of any collection of convex sets is convex. In particular, for any subset T of a plane there is the smallest convex set containing T, called the convex hull of T, which is the intersection of all convex sets containing T. Convex hull of a finite set T is a convex polygon whose vertices belong to T.

Let now S be a finite set of points, let P be the convex hull of S, and let T be the set of vertices of P, so T is a subset of S. Clearly P is also the convex hull of T. We claim that S and T have the same minimal circles. Indeed, if C is a minimal circle for T then there are 3 points from T on C and T is contained in the disc B bounded by C. Since B is convex, P is contained in B and hence S is contained in B. This means that C is a minimal circle for S. Conversely, let C be a minimal circle for S and denote by B the disc bounded by C. Thus P is contained in B. Note that if a convex polygon is contained in B then all its points which are not vertices are contained inside the circle C. It follows that the points from S which are on C must belong to T. Thus C contains three points from T and therefore C is a minimal circle for T.

For every minimal circle C of T choose three points from T which are on C. These three points form a triangle Δ_C.

Suppose now that C_1 and C_2 are two different minimal circles for T. The polygon P is contained in the discs B_1, B_2 bounded by C_1, C_2 respectively. We claim that the circles C_1 and C_2 must intersect at two distinct points. If the circles C_1 and C_2 were disjoint, then either the discs B_1 B_2 would be disjoint or one of them would be in the interior of the other, which would mean that the polygon P is in the interior of one of the discs, which is not possible (as at least three vertices of P must be on the boundary of the disc). Similar argument shows that the circles C_1 and C_2 can not be tangent. Thus C_1 and C_2 intersect at two points X and Y. The intersection of the discs D_1, D_2 is a "lense" whose boundary consists of two arcs with ends X, Y: one arc is a part of C_1 and the other arc is a part of C_2. These arcs are on opposite sides of the line XY. It follows that the points in T on C_1 are on different side of the line XY than the points of T on C_2. It follows that the triangles Δ_{C_1} and Δ_{C_2} are on opposite sides of XY, so they are either disjoint, or share a vertex, or share a side (which then coincides with $\overline{XY}$).

Assume that T has at least 4 points. Pick a minimal circle C_0 of T. Then there are two vertices A, B of the triangle Δ_{C_0} such that $\overline{AB}$ is a diagonal of P but not a side of P. This diagonal divides P into two convex polygons P_1 and P_2: one with $k \geq 3$ vertices and the other with $|T| + 2 - k$ vertices. Then for any minimal circle C of T, the triangle Δ_C must be entirely on one side of the diagonal $\overline{AB}$ (since Δ_C and Δ_{C_0} either coincide (if $C = C_0$), or are disjoint, or share a vertex, or share a side). It follows that Δ_C is either contained in P_1 or in P_2, i.e. C is either a minimal circle for the set of vertices of P_1 or a minimal circle for the set of vertices of P_2. It follows that that T has at most $M(k) + M(|T| + 2 - k)$ minimal circles.

At this point we are ready to prove that $M(n) \leq n-2$ for every n. We already know it for $n = 3$. Suppose that $n > 3$ and that every set with $k < n$ points has at most $k - 2$ minimal circles. Consider any set S with n points and let T be the subset of S consisting of vertices of the convex hull of S. Recall than S and T have the same minimal circles. Thus, if $|T| < |S| = n$, then S has at most $M(|T|) \leq |T| - 2 \leq n-2$ minimal circles by the inductive assumption. Suppose that $S = T$, so $|T| = n > 3$. According to our discussion above, the set S has at most $M(k) + M(|T| + 2 - k) = M(k) + M(n + 2 - k)$ minimal circles for some $3 \leq k < |T| = n$. By inductive assumption, $M(k) \leq k - 2$ and $M(n + 2 - k) \leq n - k$ (note that $n + 2 - k < n$), thus S has at most $(k - 2) + n - k = n - 2$ minimal circles. Since S was arbitrary,

$M(n) \leq n - 2$.

To complete the solution it suffices to construct for each $n \geq 3$ a convex n-gon whose set S_n of vertices has at least $n - 2$ minimal circles (then it must have exactly $n - 2$ minimal circles). Our construction is inductive. For $n = 3$ any triangle works. Suppose that the set S_n of vertices of some convex n-gon has at least $n - 2$ minimal circles $C_1, \ldots, C_{n-2}$. Pick a side $\overline{AB}$ of this n-gon. All vertices of the $n-$gon are on one side of the line AB. Let H be the opposite side of this line. The midpoint M of $\overline{AB}$ is in the interior of all the circles C_i. Let V be a vertex of the $n-$gon different from A and B. Note that if N is any point in H (i.e. on opposite side of AB than V) then the circumcircle of the triangle ABN contains V in its interior if and only if $\angle ANB + \angle AVB > \pi$ (i.e. the sum of these two angles is bigger than 180 degrees). We can take N on the perpendicular bisector of $\overline{AB}$ close enough to M so that N is in the interior of all the circles C_i and the angle $\angle ANB$ is as close to π as we wish, so in particular bigger than all the angles $\pi - \angle AVB$ where V is a vertex of our n-gon. Then the circumcirlcle of the triangle ANB is a minimal circle of the set $S_n \cup \{N\}$ and each of the circles C_i is also a minimal circle of this set. Thus we constructed a set $S_{n+1} = S_n \cup \{N\}$ which consists of vertices of a convex $(n+1)$-gon and which has at least $n - 1 = (n + 1) - 2$ minimal circles.