**Problem 4.** Let $f(x)$ be a polynomial of degree $n$ with $n$ distinct real roots. Prove that the polynomial

$$f(x)f''(x) - (f'(x))^2$$

has no real roots.

**Solution.** Let us write $W(f) = f(x)f''(x) - (f'(x))^2$ (some authors call $W(f)$ *the Wronskian* of the polynomial f). Since $f$ has degree $n$ and $n$ distinct real roots, there are real numbers $r_1 < r_2 < \ldots < r_n$ such that $f(x) = c(x - r_1)(x - r_2) \ldots (x - r_n)$ for some constant $c \neq 0$. Clearly no $r_j$ is a root of $W(f)$ (as otherwise $f$ and $f'$ would have a common root $r_j$ and $(x - r_j)^2$ would divide $f$, which is false). The product rule for differentiation easily implies that

$$f'(x) = \frac{f(x)}{x - r_1} + \frac{f(x)}{x - r_2} + \ldots + \frac{f(x)}{x - r_n}.$$

For $x$ different from the roots of $f$, this is the same as

$$\frac{f'(x)}{f(x)} = \frac{1}{x - r_1} + \frac{1}{x - r_2} + \ldots + \frac{1}{x - r_n}.$$

Taking the derivative of both sides yields

$$\frac{W(f)}{f^2(x)} = \frac{-1}{(x - r_1)^2} + \frac{-1}{(x - r_2)^2} + \ldots + \frac{-1}{(x - r_n)^2} = -\left(\frac{1}{(x - r_1)^2} + \frac{1}{(x - r_2)^2} + \ldots + \frac{1}{(x - r_n)^2}\right).$$

For any $x$ different from the roots of $f$, the right hand side of the last equation is negative (as any sum of squares of non-zero real numbers is positive), hence $W(f) \neq 0$ for any $x$.

**Problem.** Prove that $W(fg) = f^2W(g) + g^2W(f)$. Use it to give an inductive solution of the problem.