Problem 3. Let $s(n) = \sum_{j=1}^{n} \binom{n}{j} \frac{1}{j}$ and $f(n) = \frac{2^{n+1}}{n}$. Prove that

$$\lim_{n\to\infty} n \left(\frac{s(n)}{f(n)} - 1 \right)$$

exists and find its value.

Solution. We present first our original solution.

Lemma 0.1 $s(n) = \sum_{j=1}^{n} \frac{2^j - 1}{j}$.

Proof. Let $h(x) = \sum_{j=1}^{n} \binom{n}{j} \frac{x^j}{j}$, so $h(1) = s(n)$. The derivative of $h(x)$ is

$$h'(x) = \sum_{j=1}^{n} \binom{n}{j} x^{j-1} = \frac{1}{x} \sum_{j=1}^{n} \binom{n}{j} x^j = \frac{(x+1)^n - 1}{x} = \sum_{j=0}^{n-1} (1+x)^j.$$

Since $h(0) = 0$, integrating this formula yields

$$h(x) = \sum_{j=0}^{n-1} \frac{(1+x)^{j+1} - 1}{j}.$$

Taking $x = 1$ gives us the lemma.

Lemma 0.2 *The inequality* $2^{n+2} \geq n(n-1)$ *holds for all natural numbers* n.

Proof. For $n = 1, 2, 3$ the inequality is clear. Assume that it holds for some $n \geq 3$. Then

$$2^{(n+1)+2} = 2 \cdot 2^{n+2} \geq 2n(n-1) \geq n(n+1)$$

as $2(n-1) \geq n + 1$ for $n \geq 3$. Thus the lemma follows by induction.

Lemma 0.3 *The inequality*

$$s(n) \geq \frac{2^{n+1}}{n-1}$$

holds for all $n \geq 7$.

Proof. Let c be such that

$$s(k) + c \geq \frac{2^{k+1}}{k-1}$$

for some k. We claim that

$$s(n) + c \geq \frac{2^{n+1}}{n-1}$$

for all $n \geq k$. To justify this we proceed by induction. Assuming that it holds for some $n \geq k$ we have

$$s(n+1) + c = s(n) + \frac{2^{n+1} - 1}{n+1} + c \geq \frac{2^{n+1}}{n-1} + \frac{2^{n+1} - 1}{n+1}.$$

It suffices to show that

$$\frac{2^{n+1}}{n-1} + \frac{2^{n+1} - 1}{n+1} \geq \frac{2^{n+2}}{n},$$

which is equivalent to

$$2^{n+1}(n^2 + n) + (2^{n+1} - 1)(n^2 - n) \geq 2^{n+2}(n^2 - 1),$$

i.e.

$$2^{n+2} \geq n^2 - n = n(n-1)$$

which is true by Lemma 0.2. A direct computation shows that $c = 0$ works for $k = 7$.

Lemma 0.4 *Let $d > 1$. The inequality*

$$s(n) \leq \frac{2^{n+1}}{n}\left(1 + \frac{d}{n}\right)$$

holds for all sufficiently large n.

Proof. Since $d > 1$, we have $2d/(d+1) > 1$ and therefore there is N_1 such that $(n+2)/n \leq 2d/(d+1)$ for all $n > N_1$. Suppose that $k > N_1$ and c is such that $s(k) - c \leq \frac{2^{k+1}}{k}\left(1 + \frac{d}{k+1}\right)$ (for example, we could take $c = s(k)$). We claim that

$$s(n) - c \leq \frac{2^{n+1}}{n}\left(1 + \frac{d}{n+1}\right)$$

holds for all $n \geq k$. To justify this we proceed by induction. Assuming that it holds for some $n \geq k$ we have

$$s(n+1) - c = s(n) - c + \frac{2^{n+1}-1}{n+1} \leq \frac{2^{n+1}}{n}\left(1 + \frac{d}{n+1}\right) + \frac{2^{n+1}-1}{n+1}.$$

It suffices to show that

$$\frac{2^{n+1}}{n}\left(1 + \frac{d}{n+1}\right) + \frac{2^{n+1}-1}{n+1} \leq \frac{2^{n+2}}{n+1}\left(1 + \frac{d}{n+2}\right),$$

which is equivalent to

$$\frac{2^{n+1}}{n} - \frac{2^{n+1}}{n+1} + \frac{2^{n+1}d}{n(n+1)} - \frac{1}{n+1} \leq \frac{2^{n+2}d}{(n+1)(n+2)},$$

i.e.

$$\frac{2^{n+1}(1+d)}{n} - 1 \leq \frac{2^{n+2}d}{n+2}.$$

It suffices to show that

$$\frac{2^{n+1}(1+d)}{n} \leq \frac{2^{n+2}d}{n+2}$$

which is true by definition of N_1 and the fact that $n \geq N_1$. Thus we proved that

$$s(n) \leq c + \frac{2^{n+1}}{n}\left(1 + \frac{d}{n+1}\right) = \frac{2^{n+1}}{n}\left(1 + \frac{d}{n+1} + \frac{nc}{2^{n+1}}\right).$$

It remains to note that

$$\frac{d}{n+1} + \frac{nc}{2^{n+1}} \leq \frac{d}{n}$$

for all sufficiently large n.

We are ready now to complete the solution of our problem. By the last two lemmas, given any $\epsilon > 0$ there is N_ϵ such that for all $n > N_\epsilon$ we have $\frac{n}{n-1} \leq s(n)/f(n) \leq 1 + (1+\epsilon)/n$. It follows that

$$0 \leq n\left(\frac{s(n)}{f(n)} - 1\right) - 1 \leq \epsilon$$

for all $n \geq N_\epsilon$. This means that the limit $\lim_{n\to\infty} n\left(\frac{s(n)}{f(n)} - 1\right) = 1$.

Second solution. This solution has been provided by Prof. Anton Schick. It is purely probabilistic and it is a great illustration of the versatility of probabilistic methods. We have

$$\frac{s(n)}{f(n)} = \sum_{j=1}^{n} \binom{n}{j} \frac{n}{2j}\frac{1}{2^n} = \sum_{j=0}^{n} \binom{n}{j} \min\left(\frac{n}{2}, \frac{n}{2j}\right)\frac{1}{2^n} - \frac{n}{2^{n+1}} = E\Big(\min\left(\frac{n}{2}, \frac{n}{2X_n}\right)\Big) - \frac{n}{2^{n+1}},$$

where $E(Z)$ denotes the expected value of a random variable Z and X_n is a random variable with the Binomial$(n, 1/2)$ distribution (i.e. X_n is the number of heads in n throws of a fair coin). For any property P we denote by $\mathbf{1}[P(Z)]$ the random variable which is 1 if $P(Z)$ is true and 0 otherwise. For example, $\mathbf{1}[|Z| \le \delta]$ is the random variable which is 1 when $|Z| \le \delta$ and 0 otherwise.

Set $Y_n = 2X_n/n - 1$ and note that $-Y_n$ and Y_n have the same distribution (since the probability $Y_n = a$ is the same as the probability that $X_n = (a+1)n/2$, which is the same as the probability that $X_n = n - (a+1)n/2 = n(1-a)/2$, which is the same as the probability that $Y_n = -a$). This implies that $E(Y_n^a \mathbf{1}[|Y_n| \le \delta]) = 0$ for all odd integers a. We will need some information on the higher moments of Y_n, namely that $E(Y_n^2) = 1/n$, $E(Y_n^4) \le 3/n^2$ and $E(Y_n^6) \le 15/n^3$ (see an exercise below for a hint how to prove it).

Let $0 < \delta < 1$ and $n > 4$. Note that $\min\left(\frac{n}{2}, \frac{1}{1+Y_n}\right) = \frac{1}{1+Y_n}$ except when $Y_n = -1$. Thus

$$E\Big(\min\left(\frac{n}{2}, \frac{n}{2X_n}\right)\Big) - 1 = E\Big(\min\left(\frac{n}{2}, \frac{1}{1+Y_n}\right) - 1\Big) =$$

$$= E\Big(\Big(\frac{1}{1+Y_n} - 1\Big)\mathbf{1}[|Y_n| \le \delta]\Big) + E\Big(\min\left(\frac{n}{2}, \frac{1}{1+Y_n}\right) - 1)\mathbf{1}[|Y_n| > \delta]\Big)$$

Since $\min\left(\frac{n}{2}, \frac{1}{1+Y_n}\right) \le \frac{n}{2}$, we have

$$\Big|E\Big(\Big(\min\left(\frac{n}{2}, \frac{1}{1+Y_n}\right) - 1\Big)\mathbf{1}[|Y_n| > \delta]\Big)\Big| \le \frac{n}{2}P(|Y_n| > \delta) \le \frac{n}{2}\frac{E(Y_n^6)}{\delta^6} \le \frac{15}{2n^2\delta^6}.$$

We used here the simple observation that if Z is a non-negative random variable and the probability that $Z > a$ is P then $E(Z) > aP$ (applied to $Z = Y_n^6$ and $a = \delta^6$). Furthermore, using the identity $\frac{1}{1+x} - 1 = -x + x^2 - x^3 + \frac{x^4}{1+x}$ and the observation we made earlier that $E(Y_n^a \mathbf{1}[|Y_n| \le \delta]) = 0$ for all odd integers a, we get

$$E\Big(\Big(\frac{1}{1+Y_n} - 1\Big)\mathbf{1}[|Y_n| \le \delta]\Big) = E\Big(\Big(-Y_n + Y_n^2 - Y_n^3 + \frac{Y_n^4}{1+Y_n}\Big)\mathbf{1}[|Y_n| \le \delta]\Big) =$$

$$E(Y_n^2 \mathbf{1}[|Y_n| \le \delta] + E\Big(\frac{Y_n^4}{1+Y_n}\mathbf{1}[|Y_n| \le \delta]\Big) = E(Y_n^2) - E(Y_n^2 \mathbf{1}[|Y_n| > \delta] + E\Big(\frac{Y_n^4}{1+Y_n}\mathbf{1}[|Y_n| \le \delta]\Big) =$$

$$\frac{1}{n} - E(Y_n^2 \mathbf{1}[|Y_n| > \delta] + E\Big(\frac{Y_n^4}{1+Y_n}\mathbf{1}[|Y_n| \le \delta]\Big).$$

Finally we have

$$E(Y_n^2 \mathbf{1}[|Y_n| > \delta]) \le \frac{E(Y_n^4)}{\delta^2} = \frac{3}{n^2\delta^2}$$

(since $Y_n^2 \mathbf{1}[|Y_n| > \delta] \le Y_n^2 \cdot Y_n^2/\delta^2$) and

$$E\Big(\frac{Y_n^4}{1+Y_n}\mathbf{1}[|Y_n| \le \delta]\Big) \le \frac{E(Y_n^4)}{1-\delta} \le \frac{3}{(1-\delta)n^2}.$$

Combining the above shows that

$$n\left(\frac{s(n)}{f(n)} - 1\right) = 1 + t(n)$$

where $|t(n)| \le C/n$ for some $C > 0$. It follows that $t(n)$ tends to 0 as n tends to infinity. In other words, the limit is question exists and it is equal to 1.

Exercise. Let R be the random variable which assumes values 1 and -1 with probability 1/2 each. Prove that $n \cdot Y_n$ has the same distribution as $R_1 + \ldots + R_n$, where R_i are independent random variables with the same distribution as R. Use this to compute $E(Y_n^k)$ for $k = 2, 4, 6$. Hint: $E(R_1^{m_1} R_2^{m_2} \ldots R_n^{m_n}) = 0$ unless all m_i are even, in which case it is equal to 1.

Exercise. Modify solution 2 to compute the limit

$$\lim_{n\to\infty} n\left(n\left(\frac{s(n)}{f(n)} - 1\right) - 1\right).$$

Third solution. This solution is a modification of the solution submitted by Prof. Vladislav Kargin. The solution starts with the following observation:

$$\frac{s(n)}{f(n)} = \frac{1}{2^{n+1}} \frac{n}{n+1} \sum_{j=1}^{n} \binom{n+1}{j+1} \frac{j+1}{j} = S_1 + S_2,$$

where

$$S_1 = \frac{1}{2^{n+1}} \frac{n}{n+1} \sum_{j=1}^{n} \binom{n+1}{j+1} \quad \text{and} \quad S_2 = \frac{1}{2^{n+1}} \frac{n}{n+1} \sum_{j=1}^{n} \binom{n+1}{j+1} \frac{1}{j}.$$

Now

$$\sum_{j=1}^{n} \binom{n+1}{j+1} = 2^{n+1} - 1 - (n+1) = 2^{n+1} - n$$

which easily implies that $\lim_{n\to\infty} n(S_1 - 1) = -1$. We need to compute $\lim_{n\to\infty} nS_2$. which is the same as

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{j=1}^{n} \binom{n+1}{j+1} \frac{n}{j}.$$

We will show that this limit is equal to 2, which implies that the limit in question is $-1 + 2 = 1$. Since

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{j=1}^{n} 2\binom{n+1}{j+1} = 2,$$

it suffices to show that

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{j=1}^{n} \binom{n+1}{j+1} \left(\frac{n}{j} - 2\right) = \lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{j=1}^{n} \binom{n+1}{j+1} \frac{n-2j}{j} = 0.$$

We split the sum in the last formula into two parts:

$$\frac{1}{2^{n+1}} \sum_{j=1}^{n} \binom{n+1}{j+1} \frac{n-2j}{j} = \frac{1}{2^{n+1}} \sum_{|j-\frac{n}{2}|\le\alpha} \binom{n+1}{j+1} \frac{n-2j}{j} + \frac{1}{2^{n+1}} \sum_{|j-\frac{n}{2}|>\alpha} \binom{n+1}{j+1} \frac{n-2j}{j}$$

where $\alpha = \alpha(n) < n/2$ is some number to be specified soon. Note that the condition $|\frac{n}{2} - j| \le \alpha$ implies that $j \ge \frac{n}{2} - \alpha$ and therefore

$$\left|\frac{n-2j}{j}\right| = \frac{2|\frac{n}{2} - j|}{j} \le \frac{2\alpha}{\frac{n}{2} - \alpha} = \frac{4}{\frac{n}{\alpha} - 2}.$$

It follows that

$$\left|\frac{1}{2^{n+1}} \sum_{|j-\frac{n}{2}|\le\alpha} \binom{n+1}{j+1} \frac{n-2j}{j}\right| \le \frac{1}{2^{n+1}} \frac{4}{\frac{n}{\alpha} - 2} \sum_{|j-\frac{n}{2}|\le\alpha} \binom{n+1}{j+1} \le \frac{4}{\frac{n}{\alpha} - 2}.$$

If we choose $\alpha = \alpha(n)$ so that $\lim_{n\to\infty} n/\alpha(n) = \infty$ then we have

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{|j-\frac{n}{2}|\le\alpha(n)} \binom{n+1}{j+1} \frac{n-2j}{j} = 0.$$

To estimate the second part $\frac{1}{2^{n+1}}\sum_{|j-\frac{n}{2}|>\alpha}\binom{n+1}{j+1}\frac{n-2j}{j}$ we will use the following estimate of the binomial coefficients:

Lemma. There is a constant $D>0$ such that if $|j-\frac{n}{2}|>50\sqrt{n\ln n}$ then

$$\binom{n}{j}\leq\frac{2^nD}{\sqrt{n}n^{5000}}.$$

It follows that if $\alpha=\alpha(n)=50\sqrt{n\ln n}$ and $|j-\frac{n}{2}|>\alpha(n)$ then $\binom{n+1}{j+1}\leq n\binom{n}{j}\leq\frac{2^nD\sqrt{n}}{n^{5000}}$ and $|n-2j|/j\leq n$ so

$$\left|\frac{1}{2^{n+1}}\sum_{|j-\frac{n}{2}|>\alpha(n)}\binom{n+1}{j+1}\frac{n-2j}{j}\right|\leq\frac{1}{2^{n+1}}\cdot n\cdot\frac{2^nD\sqrt{n}}{n^{5000}}\cdot n=\frac{D\sqrt{n}}{2n^{4998}}$$

and therefore

$$\lim_{n\to\infty}\frac{1}{2^{n+1}}\sum_{|j-\frac{n}{2}|>\alpha(n)}\binom{n+1}{j+1}\frac{n-2j}{j}=0.$$

Since $n/50\sqrt{n\ln n}$ tends to infinity, we can conlude that

$$\lim_{n\to\infty}\frac{1}{2^{n+1}}\sum_{j=1}^{n}\binom{n+1}{j+1}\frac{n-2j}{j}=0,$$

which completes our solution.

For the proof of the Lemma the reader should consult page 66 of the following book (which is a book worth reading regardless of our problem):

Asymptotia by Joel Spencer, Student Mathematical Library, Vol 71, published by AMS.

or consider it as a challenging exercise.