Problem 3. Given a sequence $a_1, a_2, \ldots, a_n$ of n real numbers we construct a new sequence of $n-1$ numbers as follows: first we set $b_i = \max(a_i, a_{i+1})$ for $i = 1, ..., n-1$. Then we choose randomly one index i and add 1 to b_i. This is our new sequence. After repeating this operation $n-1$ times we arrive at a single number A. Prove that if $a_1 + \ldots + a_n = 0$, then $A \geq \log_2 n$.

Here $\max(a, b)$ denotes the larger of the numbers a, b.

Solution. Our procedure assigns to a sequence $a_1, a_2, \ldots, a_n$ of n real numbers a new sequence $b_1, b_2, \ldots, b_i + 1, \ldots, b_{n-1}$ of $n-1$ real numbers, where i is a randomly chosen integer between 1 and $n-1$ and $b_j = \max(a_j, a_{j+1})$ for $j = 1, ..., n-1$. Our solution is based on the following key observation:

$$2^{b_1} + \ldots + 2^{b_i+1} + \ldots + 2^{b_{n-1}} \geq 2^{a_1} + \ldots + 2^{a_n}.$$

To justify this inequality note that $b_j \geq a_j$ and $b_j \geq a_{j+1}$ for all j. It follows that

$$2^{b_j} \geq 2^{a_j} \text{ for j= 1,..., i-1},$$

$$2^{b_i+1} = 2^{b_i} + 2^{b_i} \geq 2^{a_i} + 2^{a_{i+1}},$$

$$2^{b_j} \geq 2^{a_{j+1}} \text{ for j= i+1,..., n-1}.$$

Adding all these inequalities yields our claim.

Applying our key observation to each step of the construction we see that

$$2^A \geq 2^{a_1} + \ldots + 2^{a_n}.$$

Recall now the AM-GM inequality, which states that for any non-negative numbers $u_1, \ldots, u_m$ their arithmetic mean (AM) $\dfrac{u_1 + \ldots + u_m}{m}$ is at least as big as their geometric mean (GM) $\sqrt[m]{u_1 u_2 \ldots u_m}$, i.e.

$$\frac{u_1 + \ldots + u_m}{m} \geq \sqrt[m]{u_1 u_2 \ldots u_m}$$

(and equality holds if and only if $u_1 = \ldots = u_m$). It follows that

$$2^a \geq 2^{a_1} + \ldots + 2^{a_n} \geq n \sqrt[n]{2^{a_1} \cdot 2^{a_2} \cdot \ldots \cdot 2^{a_n}} = n 2^{(a_1 + \ldots + a_n)/n}$$

and consequently,

$$A \geq \log_2 n + \frac{a_1 + \ldots + a_n}{n}.$$

In particular, if $a_1 + \ldots + a_n = 0$ then $A \geq \log_2 n$.

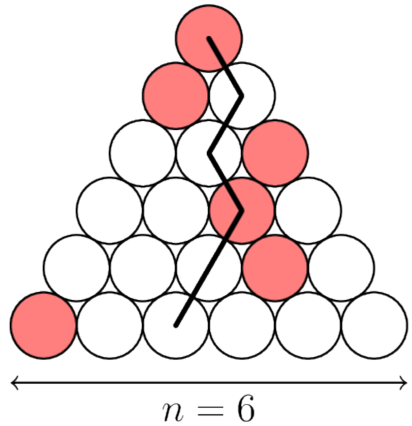
The following problems seem natural questions to ask.

Problem. Prove that if the inequality in the problem is an equality then $a_1 = \ldots = a_n = 0$ and n is a power of 2. Conversely, show that if n is a power of 2 and $a_1 = \ldots = a_n = 0$ then the equality holds for some choices of the index in each step.

Problem. Note that if $a_1 = \ldots = a_n = 0$ then A is an integer hence $A \geq \lceil \log_2 n \rceil$. Is this inequality always true? Can the equality hold?

Finally, using the ideas of this problem we suggest solving the following problem from the 2023 IMO (International Mathematical Olympiad):

Problem. Let n be a positive integer. A Japanese triangle consists of $1 + 2 + \cdots + n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots n$, the i-th row contains exactly i circles, exactly one of which is coloured red. A ninja path in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one of the two circles immediately below it and finishing in the bottom row. Below is an example of a Japanese triangle with $n = 6$, along with a ninja path in that triangle containing two red circles. In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.


$n = 6$