Problem 2. 2024 points are chosen in space so that among any 18 of these points there are two which are no farther apart than 1. Prove that there exists a ball of radius 1 which contains at least 120 of the 2024 points.

Solution. Let m be the largest number such that there exist m points among the 2024 points with the property than the distance between any two of them is larger than 1. The assumption of the problem says that $m \leq 17$. Pick such m points $P_1, \ldots, P_m$, so the distance $|P_i P_j| > 1$ for any $i \neq j$. If P is any of the remaining $2024 - m$ points, then among the $m+1$ points $P_1, \ldots P_m, P$ there must be two whose distance is at most 1 (dierctly from the definition of m). It follows that the distance between P and one of the points P_i is at most 1. Let T_i be the collection of all points whose distance to P_i is at most 1. We just observed that each of our 2024 points belongs to at least one of the sets $T_1, \ldots T_m$ (note that P_i belongs to the set T_i). This means thant $|T_1| + |T_2| + \ldots + |T_m| \geq 2024$. Consequently, there is j such that $|T_j| \geq 2024/m$. Since $m \leq 17$, we see that $|T_j| \geq 2024/17 = 179 + 1/17$. Thus T_j has at least 180 points. This means that the ball with center P_j and radius 1 contains at lest 180 of our 2024 points.

Excercise. Find a collection of 2023 points such that no ball of radius 1 contains more than 179 of the points but among any 18 of these points there are two whose distance is at most 1. .

Remark. Following Slava Kargin and Ashton Keith, we consider a graph whose vertices are the given points and two vertices are connected by an edge if and only if the distance between them does not exceed 1. According to our problem, the resulting grapgh does not have any subgraph with 18 verices and no edges between them. A set of vertices in a grapgh with no edges between them is called an **independent** subset of vertices. So our problem can be easily derived from the following result in graph theory:

A graph with M vertices and no independent set of vertices of size n must have a vertex of degree at least $M/(n-1) - 1$.

This is essentially what Ashton Keith proves by induction on n. Recall that the degree of a vertex v is the number of edges incident to v or, equivalently, the number of vertices connected by an edge to v).