Problem 1. A continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has the following property:

$$f(x) \cdot f(f(x)) = 1 \text{ for every } x \in \mathbb{R}.$$

Knowing that the largest value of f is e, prove that

$$3 + e^{-2} < \int_0^{2e} f(t)dt < 3 + e^2.$$

Show that these bounds are best possible. Here $e = 2.7128...$ is the base of natural logarithms.

Solution. The condition

$$f(x) \cdot f(f(x)) = 1 \text{ for every } x \in \mathbb{R} \tag{1}$$

means that $f(u) = 1/u$ for every u in the range of f. In particular, 0 is not in the range of f. Our first step is to determine the range of f. We are given that e is in the range of f and $f(x) \leq e$ for all x. It follows that $f(e) = 1/e$, so $1/e$ is in the range of f and $f(1/e) = e$. Thus both $1/e$ and e belong to the range of f. Since f is continuous, the intermediate value theorem tells us that any number between $1/e$ and e is also in the range of f. In other words, the closed interval $[1/e, e]$ is contained in the range of f.

Suppose now that u is in the range of f. If $u \leq 0$ then 0 would be in the range of f by the intermediate value theorem. Since we know that 0 is not in the range of f, we conclude that $u > 0$. We have $f(u) = 1/u$, so $0 < 1/u \leq e$. Thus $u \geq 1/e$. This proves that any number in the range of f is in the interval $[1/e, e]$.

Putting the above observations together, we see that the range of f is exactly the interval $[1/e, e]$. Thus f is a continuous function such that $1/e \leq f(x) \leq e$ for all x and $f(x) = 1/x$ for all $x \in [1/e, e]$. Conversely, it is straightforward to see that any continuous function f such that $1/e \leq f(x) \leq e$ for all x and $f(x) = 1/x$ for all $x \in [1/e, e]$ satisfies condition (1).

Note now that

$$\int_0^{2e} f(t)dt = \int_0^{1/e} f(t)dt + \int_{1/e}^{e} f(t)dt + \int_e^{2e} f(t)dt = \int_0^{1/e} f(t)dt + \int_{1/e}^{e} \frac{dt}{t} + \int_e^{2e} f(t)dt =$$

$$\int_0^{1/e} f(t)dt + (\ln e - \ln(1/e) + \int_e^{2e} f(t)dt = 2 + \int_0^{1/e} f(t)dt + \int_e^{2e} f(t)dt.$$

For the upper bound, note that

$$\int_0^{1/e} f(t)dt \leq \int_0^{1/e} edt = 1.$$

Since f is continuous and $f(e) = 1/e < 1$, there is $\epsilon > 0$ such that $f(x) \leq 1$ for all $x \in [e, e + \epsilon]$. We may assume that $\epsilon < e$. Thus

$$\int_e^{2e} f(t)dt = \int_e^{e+\epsilon} f(t)dt + \int_{e+\epsilon}^{2e} f(t)dt \leq \int_e^{e+\epsilon} dt + \int_{e+\epsilon}^{2e} edt = \epsilon + e(e - \epsilon) < e^2.$$

Putting these inequalities together, we see that

$$\int_0^{2e} f(t)dt < 2 + 1 + e^2 = 3 + e^2.$$

The argument for the lower bound is similar. First we have

$$\int_e^{2e} f(t)dt \geq \int_2^{2e} \frac{dt}{e} = 1.$$

Since f is continuous and $f(1/e) = e > 1$, there is $\epsilon > 0$ such that $f(x) \geq 1$ for all $x \in [1/e - \epsilon, 1/e]$. We may assume that $\epsilon < 1/e$. Thus

$$\int_0^{1/e} f(t)dt = \int_0^{1/e-\epsilon} f(t)dt + \int_{1/e-\epsilon}^{1/e} f(t)dt \geq \int_0^{1/e-\epsilon} \frac{dt}{e} + \int_{1/e-\epsilon}^{1/e} dt = \left(\frac{1}{e} - \epsilon\right)\frac{1}{e} + \epsilon > \frac{1}{e^2}.$$

It follows that

$$\int_0^{2e} f(t)dt > 2 + 1 + e^{-2} = 3 + e^{-2}.$$

It remains to show that the bounds are best possible. For every $1 > \epsilon > 0$ consider the following function:

$$f_\epsilon(x) = \begin{cases} e & \text{for } x < 1/e \\ 1/x & \text{for } x \in [1/e, e] \\ \frac{1}{e} + \frac{x-e}{\epsilon}\left(e^2 - 1\right) & \text{for } x \in [e, e + \epsilon/e] \\ e & \text{for } x > e + \epsilon/e. \end{cases}$$

It is easy to see that f_ϵ satisfies the assumptions of the problem, i.e. it is continuous, satisfies (1), and its largest value is e. We have

$$\int_0^{2e} f_\epsilon(t)dt \geq \int_0^{1/e} f_\epsilon(t)dt + \int_{1/e}^{e} f_\epsilon(t)dt + \int_{e+\epsilon/e}^{2e} f_\epsilon(t)dt = \int_0^{1/e} edt + \int_{1/e}^{e} \frac{dt}{t} + \int_{e+\epsilon/e}^{2e} edt =$$

$$1 + (\ln e - \ln(1/e) + (e - \epsilon/e)e = 3 + e^2 - \epsilon.$$

It follows that for every $a < 3 + e^2$ there is a function f which satisfies the conditions of the problem and such that

$$\int_0^{2e} f(t)dt \geq a.$$

This proves that the upper bound $3 + e^2$ can not be improved. We leave it as a straightforward exercise to write a similar argument that the lower bound $3 + e^{-2}$ can not be improved.