Problem 6. n points are chosen randomly and independently on a circle, each with the uniform distribution. What is the probability that all n points are contained in some closed semicircle?

Solution. We start with some straightforward but useful observations. We will use clockwise orientation of the circle. Let O be the center of the circle. There is a bijective correspondence between points on the circle and closed semicircles. Namely a point A of the circle determines the closed semicircle obtained by traveling on the circle from A clockwise to the other end of the diameter AO. We will call this semicircle the semicircle starting at A. Suppose now that we have n distinct points $A_1, \ldots, A_n$ contained in a semicircle starting at A. Then traveling clockwise from A along the circle we will encounter first some point A_i. Then our points will be contained in the semicircle starting at A_i and A_i is the only one among our points having this property.

Now we can solve our problem. Let the chosen points be $P_1, \ldots P_n$. Since the probability that two of them coincide is 0, we may assume that these points are pairwise distinct. Let E_i be the event that all n points are contained in the semicircle starting at P_i. According to our discussion above, these events are mutually exclusive and our points are contained in a semicircle if and only if one of the E_i happens. Thus the probability in question is equal to the sum of the probabilities of the events E_i. Note that each of the points P_j for $j \neq i$ can be on each side of the diameter P_iO with probability 1/2, and these events are independent. Thus the probability of E_i, which is the probability that all $n-1$ points are on the right side of P_iO (i.e. on the semicircle starting at P_i) is equal to $1/2^{n-1}$. Thus the probability in question is equal to $n/2^{n-1}$.

Second solution. We will look at a discrete version of our problem. Suppose that our n points are selected from the vertices $V_1, \ldots, V_{2N+1}$ of a regular $(2N+1)$-gon. Thus we have $\binom{2N+1}{n}$ possible elementary events. We need to compute the probability $p_N = P(S)$ of the event S that all n points are contained in some closed semicircle. Let S_i be the event that the selected points include V_i and are contained in the semicircle starting at V_i. As before, these events are mutually exclusive and and our points are contained in a semicircle if and only if one of the S_i happens. Thus $P(S) = P(S_1) + \ldots + P(S_{2N+1})$. Note that the elementary events in S_i are exactly those in which the remaining $n-1$ points are selected from the N vertices contained in the semicircle starting at V_i and different from V_i. Thus S_i consists of $\binom{N}{n-1}$ elementary events. It follows that

$$P(S_i) = \frac{\binom{N}{n-1}}{\binom{2N+1}{n}}$$

and

$$p_N = P(S) = (2N+1)\frac{\binom{N}{n-1}}{\binom{2N+1}{n}} = n\frac{N(N-1)\ldots(N-n+2)}{2N(2N-1)\ldots(2N-n+2)}.$$

Now, when we let N tend to infinity, the probability p_N should approach the probability in question (this is more-or-less the meaning of uniform distribution). It is clear that

$$\lim_{N\to\infty} p_N = \frac{n}{2^{n-1}}.$$

Problem. In our second solution we were working with unordered tuples of n distinct points. Carry over similar computations with ordered tuples, with or without repetitions, and see that the limit probability is the same.