Problem 5. A group of 100 friends plans a trip to a remote island to play a new exciting game. Upon arrival each of them will be given one of 199 possible nick-names to use during the game. Each player will know the nick-name of all the other players, but not her own. No communication will be possible between the players while they are on the island. However, at the end of the game each player will be asked about her nick-name. Only if at least one player answers correctly, they all will be allowed to return home. Can they prepare a strategy which will guarantee they can return?

Solution. Let N be the set of all 199 nick-names. Suppose the group can construct a function F which to each subset P of 99 elements from N assigns a subset $F(P)$ consisting of 100 elements from N so that:

1. $P \subseteq F(P)$ for every subset P of N with 99 elements.
2. every subset of 100 elements from N is equal to $F(P)$ for some P.

Suppose now that Q is the set of the 100 nick-names given to the players. Then $Q = F(P)$ for some P which consists of 99 elements from Q. The player whose nick-name is not in P will know the subset P (which consists of the nicknames given to her friends). When she computes $F(P)$, she will know Q. In particular, she will know the only nick-name in $Q - P$, i.e. her nick-name. Thus the following strategy will work: each player applies F to the set S of 99 nick-names given to her friends and uses the unique name in $F(S) - S$ in her answer. One of them will guess correctly!

It remains to to show that a function F as above exists.

First method (after Leo Kargin): Leo's solution provides a rather explicit construction of a function F with the required properties. We will order the nick-names from 1 to 199 (for example, by listing them in alphabetical order), so we will use the numbers instead of the nick-names. In other words, $N = \{1, 2, \ldots, 199\}$. Consider a subset B of N with 99 elements. Set Σ_B to be the sum of all elements in B. Let the elements not in B be $s_1 < s_2 < \ldots < s_{100}$. Adding s_i to B produces a set A_i with 100 elements. Suppose that s_i is the k_i–th smallest element of A_i. Note that between s_i and s_{i-1} there are exactly $s_i - s_{i-1} - 1$ numbers, and they all are all in B. it follows that $k_i - k_{i-1} = s_i - s_{i-1} - 1$. In other words

$$s_i - k_i = s_{i-1} - k_{i-1} + 1$$

for $i = 2, \ldots, 100$. In particular, the sequence $s_1 - k_1, s_2 - k_2, \ldots, s_{100} - k_{100}$ consists of 100 consecutive integers. Thus there is exactly one j such that $\Sigma_B + s_j - k_j$ is divisible by 100. We define $F(B) = A_j = B \cup \{s_j\}$. That F has property 1 is obvious from the definition. In order to show property 2 we define a function G which to any subset A of N with 100 elements assigns the set $G(A) \subset A$ defined as follows: remove from A the k-th smallest element, where $1 \leq k \leq 100$ is the unique integer such that 100 divides $\Sigma_A - k$ (here Σ_A is the sum of all elements in A). It is straightforward from the definitions that $G(F(B)) = B$ and $F(G(A)) = A$.

Remark. Note that the number of subsets of N with 99 elements is $\binom{199}{99}$ which is the same as the number of subsets of N with 100 elements (as $\binom{199}{100} = \binom{199}{99}$). It follows that any function F with required properties must be a bijection between subsets with 99 elements and subsets with 100 elements. Leo in his solution uses the above function G and gives a slightly different proof that G is injective (hence also surjective).

Second method (small modification of Leo's method): The function F in our first solution is still a bit cumbersome to compute. Let us instead use the following function F_1:

$$F_1(B) = G(B')'$$

where $T' = N - T$ is the complement of T in N. It is straightforward to check that F_1 is a bijection and has the required property: since $G(B') \subseteq B'$, we have $B = (B')' \subseteq G(B')' = F_1(B)$.

It turns out that the element a_B in $F_1(B) - B$ is rather easy to compute. Namely, let $A = B' = N - B$. Then a_B is the k-th smallest element in A, where k is the unique integer between 1 and 100 such that $\Sigma_A - k$ is divisible by 100. Note that $\Sigma_A + \Sigma_B = 1 + \ldots + 199 = 19900$ is divisible by 100. Thus $k = 100 - s$, where s is the reminder of Σ_B modulo 100.

This leads to a rather simple strategy for the players:

Each player knows the 99 numbers representing nick-names of her friends. The player computes the sum s of these numbers modulo 100 (so $0 \leq s < 100$) and sets $k = 100 - s$. Then she uses the k-th smallest of the remaining 100 numbers as her answer.

Third method: This method provides a much less explicit argument for the existence of functions F with the required properties. It is based on the following well know theorem in combinatorics, which is of independent interest as it has many nice applications to various areas of mathematics.

Hall's Marriage Theorem. Suppose we have n customers looking to buy a car. Each customer likes some of the cars on the lot. A necessary and sufficient condition that each customer can buy a car he likes is the following: for every $1 \leq k \leq n$, and for any k customers, the number of cars liked by at least one of the k customers is at least k.

One of the early formulations of this result involved girls trying to marry boys they like, hence the name. This result can be stated in the language of graph theory (matchings in bipartite graphs) or combinatorics (choice functions), but we decided to use the above colloquial formulation.

Let us first see how this result can be applied to our problem. Then we will provide a proof of Hall's theorem. To this end, we will think of subsets of N with 99 elements as customers and the subsets of N with 100 elements as cars. A customer B likes a car A if $B \subseteq A$. Note that in this set-up every customer likes exactly 100 cars and every car is liked by exactly 100 customers. Consider now some k customers and suppose they like t cars in total. Let us count all pairs (B, A) where B is one of our k customers and A is a car B likes. On one hand, for each B we have exactly 100 cars B likes, so the number of such pairs is $100k$. On the other hand, each of the t cars is liked by at most 100 of our customers, so the number of pairs is not grater than $100t$. Thus $100t \geq 100k$, i.e. $t \geq k$. This proves that the condition in the Hall's theorem is satisfied. There is then a one-to-one function F from customers to cars such that B likes $F(B)$ for every B. In other words, we have a one to one function form subsets of N with 99 elements to subsets of N with 100 elements such that $F(B) \subseteq B$ for every B. Since the number of subsets with 99 elements is the same as the number of subsets with 100 elements, the function F is a bijection and every set with 100 elements is equal to $F(B)$ for some B. Thus F is a function with required properties.

We end this discussion with a proof of Hall's theorem. Necessity of the condition in Hall's theorem is clear: if some k of the customers like together less than k cars, then clearly there is no way to sell each of them a car they like. To show that the condition is sufficient we proceed by induction on the number n of customers. When $n = 1$ the result is obvious. Suppose the theorem holds when the number of customers is less than n and consider the case of n customers. Our reasoning is split into two cases:

case 1: there exist $1 \leq k < n$ and a group of k customers who collectively like exactly k cars. Let A_1 be the set of our k customers and C_1 be the set of cars liked by at least one of them. Thus both A_1 and C_1 have k elements. Let A_2 be the set of remaining $n - k$ customers and C_2 be the set of remaining cars (i.e. cars not in C_1). Clearly any s customers in A_1 like in total at least s cars in C_1 (as non of them likes any cars in C_2). By the inductive assumption, the customers in A_1 can buy cars from C_1 so that each gets a car he likes. Now look at any collection of t customers in A_2. Together with the customers in A_1 they like at least $k + t$ cars. But the customer in A_1 only like the k cars in C_1, hence the t customers in A_2 must like at least t cars in C_2. This means that the inductive assumption can be applied to the $n - k$ customers in A_2 and the cars in C_2, hence customers from A_2 can buy cars from C_2 so that each gets a car he likes. Thus the result holds for our n customers.

case 2: for every $1 \leq k < n$, any group of k customers likes at least $k+1$ cars in total. In this case pick any customer and sell him a car he likes. The remaining $n - 1$ customers and the remaining cars satisfy the condition of Hall's theorem. By the inductive assumption, we can sell them cars as required.

This completes our proof of Hall's theorem.

Problem. Give a direct proof that the strategy in our second method works.

Problem. Let N be a set with n elements. Given $k < n/2$, construct an explicit injective function F from subsets of N with k elements to subsets of N with $k+1$ elements such that $B \subseteq F(B)$ for all B. Given $k > n/2$, construct an explicit injective function F from subsets of N with k elements to subsets of N with $k-1$ elements such that $F(B) \subseteq B$ for all B.